Lecture 1 : Monday March 30th

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KEY CONCEPTS : OPEN BALL, OPEN SET, CONTINUOUS SURFACE
KNOW WHETHER A GIVEN SET IS OPEN

1.1 Functions of several variables

Recall that $\mathbb{R}^n$ denotes the set of all n-tuples, or n-dimensional vectors, of real numbers. For $n = 1$ we have the usual real line, and for $n = 2$ we have the plane. In this course, a function $f : \mathbb{R}^n \to \mathbb{R}^m$ assigns to each vector in $\mathbb{R}^n$ a vector in $\mathbb{R}^m$. This is called a function of n variables, since the domain consists of n-dimensional vectors. So, for example, the function

$$f : \mathbb{R}^2 \to \mathbb{R}^3 \quad \text{defined by} \quad f(x, y) = (x, x + y, x/y)$$

is a function of two variables. Sometimes we will refer to a function $f : \mathbb{R}^n \to \mathbb{R}^m$ for $m > 1$ as a vector-valued function or a vector field, especially later in the course.

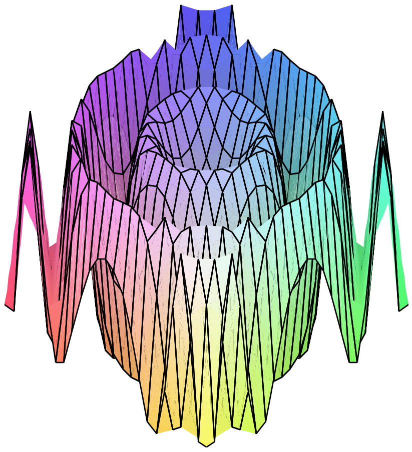
In certain instances, the domain of a function of several variables is restricted to be a subset of $\mathbb{R}^n$. For example, the domain of the function above is

$$\{(x, y) \in \mathbb{R}^2 : y \neq 0\}.$$

Almost all concepts in single variable calculus are underpinned by the notion of the limit. The same is true for several variables, where we will be studying continuity and differentiability as well as integration based on the definition of limits. As a matter of notation, we still write $f(x) = a$ for a function $f : \mathbb{R}^n \to \mathbb{R}^m$ where x and a are vectors (underlining and boldfacing becomes tedious, so vectors are represented by plain letters where it is clear). We will stick to the convention that x represents $(x_1, x_2, \ldots, x_n)$ if $x \in \mathbb{R}^n$. We will mostly concentrate on the case of functions $f : \mathbb{R}^n \to \mathbb{R}$ to start with: so a function $f : \mathbb{R}^n \to \mathbb{R}$ assigns a real number to each n-dimensional vector.

1.2 Limits and continuity

To speak of continuity of a function at a point we need the notion of a limit. For a function $f : \mathbb{R} \to \mathbb{R}$ we have precise definitions of the limit $\lim_{x \to a} f(x)$. Moreover the function is continuous at a if this limit equals $f(a)$. Intuitively, we look at the graph of the function and the function is continuous at a if it is defined at a and there is no “jump” at a. The graph of a function $f : \mathbb{R}^2 \to \mathbb{R}$ is the set of all the points $(x, y, f(x, y))$ in $\mathbb{R}^3$. If we think of $f(x, y)$ as the height of the function at point (x, y), the graph geometrically is a surface. With a bit of artistic talent we can draw surfaces, although software packages are very good at representing these graphs. For example, the surface $f(x, y) = \sin(x^2 + y^2)$ is shown below using Maple:



$$f(x, y) = \sin(x^2 + y^2)$$

It is intuitively obvious that $f(x, y) = \sin(x^2 + y^2)$ is a continuous surface. So now we come to the definition of limits and continuity for such functions.

The distance between two points $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$ is the Euclidean distance, defined by

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}.$$

This is a natural generalization of the distance between two points in $\mathbb{R}$ and in $\mathbb{R}^2$.

An open ball of radius ϵ with center x in $\mathbb{R}^n$ is the set of points $\{y \in \mathbb{R}^n : d(x, y) < \epsilon\}$. So in $\mathbb{R}$, this is just an interval of length 2ϵ (in other words ϵ on each side of x) whereas in $\mathbb{R}^2$ it is a disc. A set $U \subset \mathbb{R}$ is called open if for every point $x \in U$, there is a ball centered at x contained in U. This definition is crucial to defining the limit, so let's look at an example.

The set $U = \{(x, y) : x^2 + y^2 < r^2\}$ is an open ball of radius r centered at $(0, 0)$. It is an open set, since for every $u \in U$, any ball centered at u whose radius is less than the distance from u to the circle $x^2 + y^2 = r^2$ is entirely contained in U. So this meets the requirement that around every point in U, there is a ball entirely contained in U. The set $\{(x, y) : x > y\}$ is also an open set, as is the whole plane, $\mathbb{R}^2$. A single point, such as the origin is not an open set: no open ball around the origin is contained in a single point. The set of points on a line like $y = x$ is not an open set in $\mathbb{R}^2$, and neither is the set $\{(x, y) : x \leq 0\}$ – the left half of the plane. Again, around any point on the y-axis, every open ball around that point will contain a point in the left half of the plane and a point not in the left half of the plane. Now that we have the notion of open sets, we can define limits.