Lecture 10 - Wednesday April 22nd

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10.1 Constrained Optimization

A set $U \subset \mathbb{R}^n$ is called closed if the complement $\mathbb{R}^n \backslash U$ is open. For example, the set $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ is a closed set since $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 > 1\}$ – its complement – is open. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function and let U be a closed subset of the domain of f so that f is continuous on U. In this section we use the second derivative methods we have so far developed to find global maxima and minima of f on U.

Definition.

> Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function and let U be a subset of the domain of f. A point $x \in U$ is a global or absolute maximum of f on U if $f(x) \geq f(y)$ for all $y \in U$, and a global or absolute minimum of f on U if $f(x) \leq f(y)$.

The following theorem guarantees the existence of absolute minima and maxima on closed sets.

Theorem.

> Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function and let U be a closed subset of the domain of f so that f is continuous on U. Then f has a global maximum and a global minimum on U.

The global extremes guaranteed by this theorem need not be unique – f may have many global maxima and minima on U. With the assumption $f \in C^2(U)$, we can use the second derivative test on U to find these global extremes.

10.2 Examples

Example 1. Find the points on the surface $z = 1/x$ closest to the origin.

Solution. We want to minimize $\sqrt{x^2 + y^2 + z^2}$ – the distance from (x, y, z) to the origin – subject to the equation $z = 1/x$. So we want to minimize $\sqrt{x^2 + y^2 + 1/x^2}$. It is equivalent to minimizing $f(x, y) = x^2 + y^2 + 1/x^2$. We can clearly assume $x \neq 0$. Now since $f \in C^2(\mathbb{R}^2 \backslash \{0\})$, we can find all local extremes of f on U using the second derivative test. First the critical points:

$$f_x = f_y = 0 \Rightarrow 2x - 2/x^3 = 0 \text{ and } 2y = 0 \Rightarrow x = \pm 1 \text{ and } y = 0.$$

So the critical points are $(1, 0)$, $(-1, 0)$, and note they are both in U.

$$H_f(x, y) = \begin{pmatrix} 2 + 6/x^4 & 0 \\ 0 & 2 \end{pmatrix}.$$

At the critical points, the Hessian matrix is clearly positive definite, so we conclude that both $(1,0)$ and $(-1,0)$ are local minima. The distance from (x,y,z) to the origin in each case is $\sqrt{2}$.

Example 2. Consider the last example, but with the restriction $2 \leq x^2+y^2 \leq 8$. This set U is closed, and since the critical points found before are not in U, they are not valid extremes of f on U. Now there are no critical points of f inside U. This means that f has no extremes on the interior of U, and now we have to check the boundary of U. The boundary consists of the circles $x^2+y^2=2$ and $x^2+y^2=8$. On the boundary $x^2+y^2=2$, the function $f(x,y)$ becomes $2+1/x^2$. So we have to minimize $2+1/x^2$ subject to $|x| \leq \sqrt{2}$. Clearly $2+1/x^2$ is minimized when $|x|$ is as large as possible, namely $x=\pm\sqrt{2}$. So on this boundary of U we obtain the value $\sqrt{2+1/2}=\sqrt{5/2}$. On the boundary $x^2+y^2=8$, we get $\sqrt{8+1/8}=\sqrt{65/8}$. Therefore the global minimum distance to the origin on the surface $z=1/x$ subject to $2 \leq x^2+y^2 \leq 8$ is $\sqrt{5/2}$, attained by the point $(\sqrt{2},0,1/\sqrt{2})$ and the point $(-\sqrt{2},0,-1/\sqrt{2})$.

This example illustrates the general method. A boundary point of a set U is a point x such that every open ball containing x contains a point in U and a point not in U. For example, if $U=\{(x,y): x^2+y^2<1\}$ then every point on the circle $x^2+y^2=1$ is a boundary point of U. The boundary of U, which is denoted ∂U, is the set of all boundary points of U. For example, if $U=\{1,2,3,\dots\}$ then $\partial U = U$, and if $U=\{(x,y): x^2+y^2<1\}$ then $\partial U=\{(x,y): x^2+y^2=1\}$.

- Determine all extreme points of f on the interior of U.
  - ∘ Locate the critical points of f in the interior of U
  - ∘ Classify those points and determine the values of f at these points.
- On ∂U determine all extreme points of f viewed as a function on ∂U.
  - ∘ Locate the critical points of f on ∂U
  - ∘ Classify those points and determine the values of f at these points.

If f has no critical points in U, then it has no extremes in U which means that the only extremes of f on U are on the boundary of U.

Example 3. Find all extremes of the function $f(x,y)=x^4+y^4+4xy$ on the set U defined by $y \leq 1-x$ and $x \geq 0$ and $y \geq 0$.

Solution. The set U is a triangle and is a closed set. On the interior of this region, we compute

$$f_x = 4x^3+4y \text{ and } f_y = 4y^3+4x$$

and these are zero when $y=-x^3$ and $x=-y^3$ i.e. when $(x,y)=(-1,1)$, or $(x,y)=(0,0)$, or $(x,y)=(1,-1)$. The first and third points can be discarded since they are not in U. The origin is in U. The Hessian at the origin is easily seen to be neither positive nor negative definite (it has a zero in the top left corner) so the second derivative test fails at $(0,0)$. Nevertheless we compute $f(0,0)=0$. Now we check the boundary of U. On the lines $x=0$ and $y=0$, we clearly have maxima at $(1,0)$ and $(0,1)$, and a minimum at $(0,0)$. Now we check the line $y=1-x$ for $0 \leq x \leq 1$. There $f(x,y)$ becomes $x^4+(1-x)^4+4x(1-x)$. The critical points of this are from $4x^3-4(1-x)^3+4-8x = 8x^3-12x^2+4x=0$ which means $4x(2x^2-3x+1)=0$ and so $x=0$ or $(2x-1)(x-1)=0$. Thus $x \in \{0,1/2,1\}$. The only point we haven't already covered is $(x,y)=(1/2,1/2)$. At this point, $f(x,y)=9/8$. Now we compare all the values we found:

$$f(0,0)=0 \quad f(1,0)=f(0,1)=1 \quad f(1/2,1/2)=9/8.$$

We conclude that $(0,0)$ is a global minimum and $(1/2,1/2)$ is a global maximum on U.