Lecture 11 - Friday April 24th

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11.1 Lagrange Multipliers

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function defined on a region R described by a functional equation or constraint $g(x) = 0$ and such that $f \in C^1(R)$. Let ϕ be a function defined by

$$\phi(x, \lambda) = f(x) + \lambda g(x)$$

where λ is called a Lagrange Multiplier and is to be determined later. Then amongst all points $x \in R$ and $\lambda \in \mathbb{R}$ such that $\nabla\phi(x, \lambda) = 0$, the extreme points of f is guaranteed to appear. That this is true follows from the theory of implicit functions, and this method is referred to as the method of Lagrange Multipliers.

Example 1. Find the dimensions of the box of largest volume which can be fitted inside the sphere $x^2 + y^2 + z^2 = 1$.

Solution. We can assume the sides of the boxes are parallel to the co-ordinate axes, so if (x, y, z) is a corner of the box on the sphere with $x > 0$ and $y > 0$ and $z > 0$, then the volume of the box is $8xyz$. By Lagrange's Method, to find the maximum volume we have to consider the gradient of

$$\phi(x, y, z, \lambda) = 8xyz + \lambda(1 - x^2 - y^2 - x^2).$$

The gradient is

$$\nabla\phi = (8yz - 2\lambda x, 8xz - 2\lambda y, 8xy - 2\lambda z, 1 - x^2 - y^2 - z^2).$$

Setting this to zero we get

$$8xyz - 2\lambda x^2 = 0 \quad 8xyz - 2\lambda y^2 = 0 \quad 8xyz - 2\lambda z^2 = 0.$$

If we add these up and use $x^2 + y^2 + z^2 = 1$ we get

$$12xyz = 2\lambda \Rightarrow \lambda = 12xyz.$$

Putting this into the original equations we get $x^2 = y^2 = z^2 = 1/3$. Since we said x, y, z are all positive, we get the extreme point $x = 1/\sqrt{3}, y = 1/\sqrt{3}$ and $z = 1/\sqrt{3}$. So the volume of the largest box is $8/3\sqrt{3}$.

Example 2. Prove that

$$(x_1 + x_2 + \cdots + x_n)^n \geq n^n x_1 x_2 \ldots x_n$$

Solution. It is enough to show that the maximum value of $x_1^2 x_2^2 \ldots x_n^2$ subject to $x_1^2 + x_2^2 + \cdots + x_n^2 = r^2$ is when all the x_i are equal to $r/\sqrt{n}$. By Lagrange Multipliers, we consider

$$\phi(x, \lambda) = x_1^2 x_2^2 \ldots x_n^2 + \lambda(r^2 - x_1^2 - x_2^2 - \cdots - x_n^2).$$

Now the gradient is zero if for all i,

$$\prod_{k=1}^{n} x_k^2 = \lambda x_i^2$$

and this means $x_1^2 = x_2^2 = \cdots = x_n^2 = r^2/n$, as required.

The inequality in the last example is the well-known arithmetic mean geometric mean inequality. Another inequality which can be proved in this way is the Cauchy-Schwarz inequality:

$$\left(\sum_{i=1}^{n} x_i y_i\right)^2 \leq \left(\sum_{i=1}^{n} x_i^2\right) \cdot \left(\sum_{i=1}^{n} y_i^2\right).$$

This inequality is obvious for $n = 1$ and for $n = 2$ it is a simple exercise.