Lecture 14 - Friday May 1st

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KEY WORDS: DOUBLE INTEGRAL, RIEMANN SUM, INTEGRABLE, ITERATED INTEGRAL
KNOW DEFINITIONS AND HOW TO EVALUATE ITERATED INTEGRALS

14.1 Definition of double integrals

Recall that if $f(x)$ is a function defined for $a \leq x \leq b$, then the integral of f from a to b, when it exists, is defined as the limit

$$\int_a^b f(x)dx = \lim_{\|P\| \to 0} \sum_{I \in P} f(x_I)|I|$$

where P is a partition of $[a, b]$ into intervals, $\|P\|$ denotes the length of the longest interval $I \in P$, and x_I is an arbitrary point in I. Geometrically, if $f(x) \geq 0$ for $a \leq x \leq b$, the integral represents the area between the curve represented by $y = f(x)$ and the x-axis.

We next turn to integration of functions of two variables $f : \mathbb{R}^2 \to \mathbb{R}$. First, for simplicity, suppose $R = [a, b] \times [c, d]$ – this is the rectangle whose corners are (a, c), (a, d), (b, c) and (b, d). Let P be a partition of R into a grid of rectangles and let $\|P\|$ be the largest side length of any of the rectangles in P. Then

$$\int\int_R f dydx = \int_a^b \int_c^d f(x, y)dydx = \lim_{\|P\| \to 0} \sum_{I \in P} f(x_I, y_I)|I|$$

where (x_I, y_I) is an arbitrary point in the rectangle I and $|I|$ denotes the area of the rectangle I. The sum in the limit is referred to as a Riemann Sum and the integral is referred to as the double integral of f over R. The Riemman Sum makes intuitive sense since $f(x_I, y_I)$ is an estimate for the height of the solid enclosed by I and the part of the surface $z = f(x, y)$ above I, and so the Riemann Sum for f over R is an estimate for the volume of the solid.

If the double integral $\int\int_R f dydx$ exists then we say that f is integrable over R. A basic fact is that continuous functions are integrable on rectangles.

Theorem.

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function on a rectangle R. Then f is integrable over R.

Next we consider integrals over more general regions $D \subseteq \mathbb{R}^2$. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function defined on a region D in the plane, and suppose $D \subseteq R$ where R is a rectangle. Define a new function $g(x, y)$ by $g(x, y) = 0$ if $(x, y) \notin D$ and $g(x, y) = f(x, y)$ if $(x, y) \in D$. Then

$$\int\int_D f(x,y)dydx = \int\int_R g(x,y)dydx$$

provided g is integrable over R. The technical difficulties arise here when ∂D, the boundary of D, is complicated. However, in the case that f is continuous on $D \cup \partial D$ and ∂D is a simple closed curve, the integral over D exists. For most of the material to follow, ∂D will be a simple closed curve.

The main types of regions we consider are x-simple and y-simple regions. A region D bounded by a simple closed curve is x-simple (y-simple) if every line parallel to the x-axis (y-axis) intersects D in a line segment. For example, a circle is clearly both x-simple and y-simple. The region enclosed between $y = \sin x + 2$ and $y = \cos x - 2$ for $0 \le x \le 2\pi$ is y-simple, but not x-simple. The reason we concentrate on x-simple and y-simple curves is because the double integrals over these regions are easier to express. Indeed, if D is the region bounded between $y = g(x)$ and $y = h(x)$ for $a \le x \le b$ – this is a y-simple region – then when the integral exists,

$$\int\int_D f(x,y)dydx = \int_a^b \int_{g(x)}^{h(x)} f(x,y)dydx.$$

14.2 Evaluating double integrals

We shall see that

$$\int_a^b \int_{g(x)}^{h(x)} f(x,y)dydx = \int_a^b \left(\int_{g(x)}^{h(x)} f(x,y)dy\right) dx$$

so the double integral is actually an iterated integral. In the inside brackets, we treat x as a constant an integrate $f(x, y)$ as a function of y from $g(x)$ to $h(x)$. Then we integrate the result with respect to x from a to b.

Example 1. Evaluate

$$\int_0^1 \int_0^x x^2 e^{xy} dydx.$$

Solution. The inner integral is $\int_0^x x^2 e^{xy} dy$. Treating x as a constant, we get

$$\int_0^x x^2 e^{xy} dy = x^2 \left(\frac{1}{x} e^{x^2} - \frac{1}{x}\right) = xe^{x^2} - x.$$

Now we integrate this with respect to x:

$$\int_0^1 (xe^{x^2} - x)dx = \frac{1}{2}(e-1) - \frac{1}{2} = \frac{1}{2}e - 1.$$

This is actually the volume below x^2e^{xy} above the triangle defined by $0 \leq x \leq 1$ and $0 \leq y \leq x$.

Example 2. Evaluate

$$\int_0^1 \int_0^x e^{x^2} dydx.$$

Solution. The inner integral is $\int_0^x e^{x^2} dy$. Treating x as constant, we get $ye^{x^2}\big|_0^x$ which is xe^{x^2}. Now this can be integrated to get $\frac{1}{2}(e-1)$:

$$\int_0^1 xe^{x^2} dx = \frac{1}{2}\int_0^1 (2x)e^{x^2} dx = \frac{1}{2}\int_1^e dz = \frac{1}{2}(e-1)$$

where we used the substitution $z = e^{x^2}$.

The fact that double integrals over rectangles are actually iterated integrals follows from the definition of the double integral. If P is a partition of $R = [a,b] \times [c,d]$ into a grid of rectangles, then there exist partitions P_1 of $[a,b]$ and P_2 of $[c,d]$ such that every rectangle in P has the form $I_1 \times I_2$ with I_1 an interval in P_1 and I_2 an interval in P_2. Therefore if $F(x) = \int_c^d f(x,y)dy$, and assuming all the individual limits exist,

$$\begin{aligned}
\int\int_R f(x,y)dydx &= \lim_{\|P\|\to 0} \sum_{I \in P} f(x_I, y_I)|I| \\
&= \lim_{\|P_1\|,\|P_2\|\to 0} \sum_{I_1 \in P_1} \sum_{I_2 \in P_2} f(x_{I_1}, y_{I_2})|I_1||I_2| \\
&= \lim_{\|P_1\|\to 0} \sum_{I_1 \in P_1} |I_1| \left(\lim_{\|P_2\|\to 0} \sum_{I_2 \in P_2} f(x_{I_1}, y_{I_2})|I_2| \right) \\
&= \lim_{\|P_1\|\to 0} \sum_{I_1 \in P_1} F(x_I)|I_1| \\
&= \int_a^b \left(\int_c^d f(x,y)dy \right) dx
\end{aligned}$$

A similar argument works for more general regions. We did say that all the individual limits must exist, in particular,

$$\lim_{\|P_2\|\to 0} \sum_{I_2 \in P_2} f(x_{I_1}, x_{I_2})|I_2| = F(x_I).$$

If $f(x,y)$ is continuous on R, then for each $x \in R$, $f(x,y)$ is a continuous function. Therefore $f(x_{I_1}, y)$ is integrable and equal to $F(x_I)$. Our conclusion is that it is enough that f is continuous for the integral over R to be an iterated integral. This is certainly not true if f is allowed to be discontinuous.