Lecture 16 - Wednesday May 6th

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16.1 Improper double integrals

An improper integral is an integral $\int\int_D f dA$ such that D is an unbounded region or f is an unbounded function. For example, if $D = \mathbb{R}^2$ or $D = \{(x, y) : |x - y| \geq 1\}$, then D is unbounded. Similarly, the integral of $1/(1 - x^2 - y^2)$ over the unit disc $D = \{(x, y) : x^2 + y^2 \leq 1\}$ is also an improper integral. In this section, we shall only deal with improper integrals of functions over rectangles or simple regions D, such that f has only finitely many discontinuities in D.

We recall that if $f : \mathbb{R} \to \mathbb{R}$ is continuous at all points in $[a, b]$ except at some point $z \in [a, b]$, then we may determine the integral of f on $[a, b]$ using limits:

$$\int_a^b f(x)dx = \lim_{c \to z^-} \int_a^c f(x)dx + \lim_{c \to z^+} \int_c^b f(x)dx$$

when each of the integrals on the right exists. In other words, we take a ball around z and then let the radius of the ball shrink to zero in the limit. This deals with the single integral of a function which may be unbounded at a point z, and we can extend this to cases where the function has finitely many discontinuities. For double integrals, we proceed in a similar way. Suppose z is a discontinuity of a function f that is continuous at every other point of a bounded rectangle or simple region D. Let B_z be an open ball of radius δ containing z. Then f is continuous on $D(\delta) = D \backslash B$, which is a union of at most two simple regions. It follows that the double integral of f over D is defined and equal to an iterated integral. It turns out that

$$\int\int_D f dA = \lim_{\delta \to 0} \int\int_{D(\delta)} f dA.$$

According to Fubini's Theorem for double integrals, we can even allow f to have arbitrary discontinuities on ∂D provided that $f \geq 0$ on D:

Fubini's Theorem for improper integrals.

> If D is a bounded rectangle or simple region in the plane defined by $\{(x, y) : a \leq x \leq b, g(x) \leq y \leq h(x)\}$ and also by $\{(x, y) : c \leq y \leq d, j(y) \leq x \leq k(y)\}$, and f is a non-negative function on D which has finitely many discontinuities in the interior of D, then
>
> $$\int\int_D f dA = \int_a^b \int_{g(x)}^{h(x)} f(x, y)dydx = \int_c^d \int_{j(y)}^{k(y)} f(x, y)dxdy.$$

It is very important to note that we require f to be non-negative on D, for otherwise the theorem fails. We will not be dealing with the case where f has infinitely many discontinuities inside D.

16.2 Improper integrals for unbounded regions

We also have improper integrals where the region of integration is unbounded. For example, if f is continuous on $[a, \infty)$ then when the integral exists it is defined by

$$\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx.$$

Suppose now that f is a continuous function on an unbounded rectangle R. The definition of the double integral of f over R is

$$\int\int_R f dA = \lim_{\|P\|\to 0} \sum_{I\in P} f(x_I, y_I)|I|$$

where P is a grid partition of R. Note that the sum is infinite here since R is unbounded. It turns out that this integral is evaluated via an iterated integral too:

Proposition.

If $R = [a, \infty] \times [c, \infty]$, then when the limit exists,

$$\int\int_R f dA = \lim_{(b,d)\to\infty} \int_a^b \left(\int_c^d f(x,y)dy\right) dx = \lim_{(b,d)\to(\infty,\infty)} \int_c^d \left(\int_a^b f(x,y)dx\right) dy$$

The meaning of the limit is as follows: if $g(a, b)$ is a function then $\lim_{(a,b)\to\infty} g(a, b) = L$ if and only if for all $\varepsilon > 0$ there exists r such that if $a > r$ and $b > r$ then $|g(a, b) - L| < \epsilon$. In other words, far enough away from the origin, the function $g(a, b)$ is arbitrarily close to L. For example,

$$\lim_{(a,b)\to\infty} \frac{1}{1+a+b} = 1$$

since if we want $\frac{1}{1+a+b} < \epsilon$ it is enough to make sure that $a > 1/2\epsilon - 1/2$ and $b > 1/2\epsilon - 1/2$. On the other hand,

$$\lim_{(a,b)\to\infty} \frac{b}{1+a}$$

does not exist because we could take $b = a$ and get the limit equalling one, and $b = 2a$ and get the limit equalling two.

16.3 Examples.

Example 1. We return to the example of $\int\int_D \frac{e^y}{y} dA$ where $D = \{(x, y) : 0 \leq x \leq 1, x \leq y \leq \sqrt{x}\}$. Let's show that $f(x, y) = e^y/y$ is integrable on D. Since f is non-negative and continuous at all points of D except $(0, 0)$, we can apply Fubini's Theorm for improper integrals. Noticing that

$$D = \{(x, y) : 0 \leq y \leq 1, y^2 \leq x \leq y\}$$

is another way to write D, Fubini's Theorem says that we just have to work out

$$\int_0^1 \int_{y^2}^y \frac{e^y}{y} dx dy$$

to find the integral we want. Now we already did this integral in a preceding example, so we conclude f is integrable on D.

Example 2. Now $f(x, y) = (\sin x)/y$ and $D = [0, 2\pi] \times [0, 1]$. We cannot apply Fubini's Theorem since the function $f(x, y)$ can be negative at points in D – for example $f(3\pi/2, 1) = -1$. Moreover we saw previously that the iterated integrals of f are not equal, so f is not integrable over D.

Example 3. Find the volume under the surface $z = y/(1 - x^2 - y^2)$ lying over the quarter disc $x^2 + y^2 \leq 1$ where $x, y \geq 0$.

Solution. The function $f(x, y) = y/\sqrt{1 - x^2 - y^2}$ is non-negative and continuous at all points in the interior of the disc, so we can apply Fubini's Theorem. We will integrate on the outside with respect to x and on the inside with respect to y. Then the volume is

$$V = \int_0^1 \int_0^{\sqrt{1-x^2}} \frac{y}{\sqrt{1 - x^2 - y^2}} dy dx.$$

Let $w = x^2 + y^2$ in the inner integral. Then

$$\begin{aligned} V &= \frac{1}{2} \int_0^1 \int_{x^2}^1 \frac{1}{\sqrt{1 - w}} dw dx \\ &= \int_0^1 \sqrt{1 - x^2} dx \\ &= \frac{\pi}{4}. \end{aligned}$$

Example 4. Determine the integral of $xye^{-x^2-y^2}$ over the positive quadrant $\mathbb{R}^2$.

Solution. The given function f is continuous on $\mathbb{R}^2$, but the integral is still improper since the first quadrant Q in $\mathbb{R}^2$ is unbounded. By the proposition,

$$\int\int_Q f dA = \lim_{(a,b)\to(\infty,\infty)} \int_0^a \int_0^b xye^{-x^2-y^2} dy dx.$$

The double integral inside the limit is easily check to be equal to

$$g(a, b) = \frac{1}{4}(1 - e^{-a^2})(1 - e^{-b^2}).$$

Finally,

$$\lim_{(a,b)\to(\infty,\infty)} g(a, b) = \frac{1}{4}$$

so this is the value of $\int\int_Q f dA$.