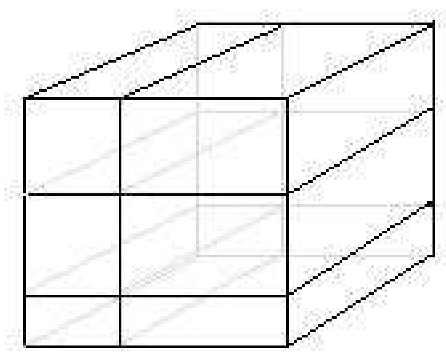
Lecture 17 - Friday May 8th

jacques@ucsd.edu

Key words: Multiple integrals
Key concepts: Know how to evaluate multiple integrals

17.1 Multiple Integrals

A rectangular parallelepiped in $\mathbb{R}^n$ is a set of the form $[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$ where $a_i < b_i$ are real numbers. We refer to this also as a Box. If B is such a box, then B consists of all vectors $(x_1, x_2, \ldots, x_n)$ such that $a_i \leq x_i \leq b_i$ for $i = 1, 2, \ldots, n$. We begin with the definition of the multiple integral of a function over a box. Let P_i be a partition of $[a_i, b_i]$ for $i = 1, 2, \ldots, n$. A grid partition of a box B consists of all boxes of the form $I_1 \times I_2 \times \cdots \times I_n$ where $I_1, I_2, \ldots, I_n$ are intervals in partitions $P_1, P_2, \ldots, P_n$ of $[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n]$. An example of such a partition is shown below:



A grid partition of a box $B = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$
$P_1 = \{[a_1, b_1]\}$, $P_2 = \{[a_2, c_2], [c_2, b_2]\}$ and $P_3 = \{[a_3, b_3]\}$

For a grid partition P of a box B, let $\|P\|$ be the volume of the largest box in P and let $|I|$ denote the volume of a box $I \in P$. Now we can define multiple integrals.

Definition of multiple integrals.

Let B be a box and let $f : \mathbb{R}^n \to \mathbb{R}$ be any function defined at every point of B. Then the multiple integral of f over B, when it exists, is defined by

$$\int\int \cdots \int_B f dV = \lim_{\|P\| \to 0} \sum_{I \in P} f(x_I)|I|$$

where P is a grid partition of B and x_I is an arbitrary point in the box $I \in P$.

We can define multiple integrals over general bounded regions D by taking a box $B \supseteq D$ and then defining $g(x) = f(x)$ for $x \in D$ and $g(x) = 0$ for $x \in B \backslash D$. Then the multiple integral of f over D is defined to be the multiple integral of g over D.

17.2 Evaluating multiple integrals

The notation dV represents the element of volume. As with single integrals, we have Fubini's Theorem for multiple integrals of continuous functions:

$$\int\int\cdot\int_B f dV = \int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_n}^{b_n} f(x_1, x_2, \dots, x_n)dx_n dx_{n-1}\dots dx_1$$

if f is a continuous function on B, and the order of integration is immaterial.

Example 1. Evaluate $\int\int\int_B f dx$ when $f(x, y, z) = x + yz$ and $B = [0, 1] \times [0, 2] \times [0, 4]$.

Solution. The given function is continuous on B, so

$$\begin{aligned}
\int\int\int_B f dx &= \int_0^1\int_0^2\int_0^4 (x + yz)dzdydx \\
&= \int_0^1\int_0^2 (xz + \frac{1}{2}yz^2)\Big|_0^4 dydx \\
&= \int_0^1\int_0^2 (4x + 8y)dydx \\
&= \int_0^1 (4xy + 4y^2)\Big|_0^2 dx \\
&= \int_0^1 (8x + 16)dx \\
&= 20.
\end{aligned}$$

We would have had the same answer integrating with respect to any of the other five orderings of the integrals.

In general it is impossible to work out multiple integrals over arbitrary regions, so we focus on simple regions. A bounded region $D \subset \mathbb{R}^n$ is simple if every line in $\mathbb{R}^n$ intersects D in a line segment. For example, the n-dimensional sphere

$$D = \{(x_1, x_2, \dots, x_n) : x_1^2 + x_2^2 + \cdots + x_n^2 \leq 1\}$$

is a simple region, and boxes are also simple regions. As in the case of double integrals, we assume simple regions are presented in the following form: if D is a simple region, then D is the set of $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ such that

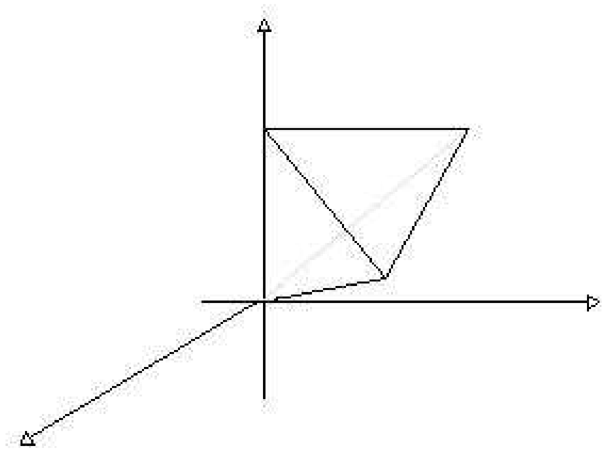
$$\begin{aligned} a \leq & \quad x_1 & \leq b \\ g_1(x_1) \leq & \quad x_2 & \leq h_1(x_1) \\ g_2(x_1, x_2) \leq & \quad x_3 & \leq h_2(x_1, x_2) \\ & \quad \vdots & \\ g_{n-1}(x_1, x_2, \ldots, x_{n-1}) \leq & \quad x_n & \leq h_{n-1}(x_1, x_2, \ldots, x_{n-1}) \end{aligned}$$

for some functions g_i, h_i such that $g_i \leq h_i$ for all $i = 1, 2, \ldots, n-1$. In the case of a multiple integral over such a region, if the integral exists and f is a continuous function on D then it is given by

$$\int\int \cdots \int_D f dV = \int_a^b \int_{g_1(x_1)}^{h_1(x_1)} \cdots \int_{g_{n-1}(x_1,x_2,\ldots,x_{n-1})}^{h_{n-1}(x_1,x_2,\ldots,x_{n-1})} f(x_1, x_2, \ldots, x_n) dx_n dx_{n-1} \ldots dx_1.$$

Furthermore, if we have a similar definition of D under a different ordering of the variables, then we can interchange the order of integration without affecting the value of the integral – this is Fubini's Theorem for multiple integrals.

Example 1. Evaluate $\int\int\int_D f dV$ when $D = \{(x, y, z) : 0 \leq x \leq y \leq z \leq 1\}$ and $f(x, y, z) = x$.



The region D

Solution. The first step is to note that D is a simple region (as shown above), and therefore we will write D in the form given above. One way of doing this is to write

$$D = \{(x, y, z) : 0 \leq x \leq 1, x \leq y \leq 1, y \leq z \leq 1\}.$$

If we do this then since f is continuous on D, we have

$$\begin{aligned}\int\int\int_D f dV &= \int_0^1 \int_x^1 \int_y^1 x dz dy dx \\ &= \int_0^1 \int_x^1 x(1-y) dy dx \\ &= \frac{1}{2}\int_0^1 x(1-2x+x^2) dx \\ &= \frac{1}{24}.\end{aligned}$$

If we had wanted to change the order of integration, to integrate with respect to x then y then z, then we would have to rewrite the definition of D:

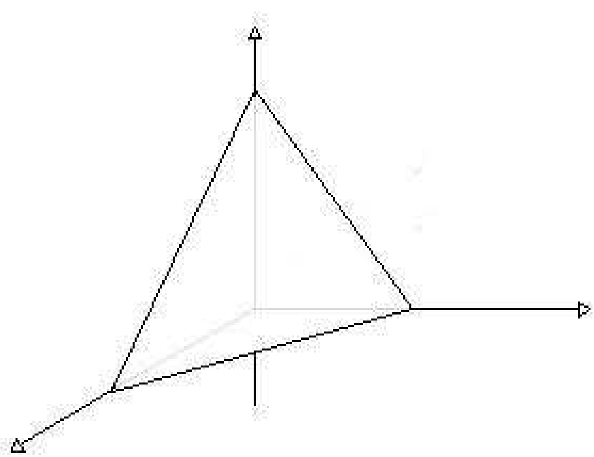
$$D = \{(x, y, z) : 0 \leq z \leq 1, 0 \leq y \leq z, 0 \leq x \leq y\}$$

in which case the integral is

$$\int\int\int_D f dV = \int_0^1 \int_0^x \int_0^y dx dy dz.$$

By Fubini's Theorem we will get the same answer as above.

Example 2. Evaluate $\int\int\int_D f dV$ when $D = \{(x, y, z) : 0 \leq x+y+z \leq 1 \text{ and } x, y, z \geq 0\}$ and $f(x, y, z) = 1$.



The region D

Solution. Since D is simple, we can write $D = \{(x, y, z) : 0 \leq x \leq 1, 0 \leq y \leq 1-x, 0 \leq z \leq 1-x-y\}$. Then since f is continuous on D,

$$\int\int\int_D f dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} 1 dV = \frac{1}{6}.$$

In fact this integral represents the volume of the pyramid shown above. We could generalize this example to determine the volume enclosed by $x_1 + x_2 + \cdots + x_n \leq 1$ where $x_1, x_2, \ldots, x_n \geq 0$. This is the volume of an n-dimensional simplex, and if we compute the integral, which is

$$\int_0^1 \int_0^{1-x_1} \int_0^{1-x_1-x_2} \cdots \int_0^{1-x_1-x_2-\cdots-x_{n-1}} 1 dx_n dx_{n-1} \ldots dx_1$$

then we get that the volume is $1/n!$.