Lecture 18 - Monday May 11th

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Key words: Change of variables, Jacobian, polar, cylindrical and spherical co-ordinates
Key concepts: Know how to change variables in integrals

18.1 Change of variables

In this section we give the change of variables theorem for multiple integrals. This is the analog of the substitution rule for integrals of functions of one variable. Suppose that we want to make the substitution $x = x(u)$ in the integral of a function f from a to b, and that for $a \leq x \leq b$, we have $c \leq u \leq d$. Then

$$\int_a^b f(x)dx = \int_c^d f(x(u))\frac{dx}{du}du.$$

The important term to remember is the dx/du term, and this term has an analog for multiple integrals.

The function $x = x(u)$ given above is actually a bijection from $[c, d]$ to $[a, b]$: this means that for every point $t \in [a, b]$, there is a unique number $u \in [c, d]$ such that $x(u) = t$. That $x(u)$ is a bijection is essential in order to make the change of integration, and it means that as a function, $x(u)$ has an inverse, denoted $x^{-1}(u)$. The inverse is itself a bijection from $[a, b]$ to $[c, d]$. For example, the function $x(u) = u^3$ is clearly a bijection from $\mathbb{R}$ to $\mathbb{R}$, since every real number has a unique real cube-root. The inverse is given by $x^{-1}(u) = u^{1/3}$ – $x(x^{-1}(u)) = u$ in this case. The function $x(u) = u^2$ is not a bijection, since $x(1) = x(-1) = 1$. To say that a function $x(u) : \mathbb{R} \to \mathbb{R}$ is a bijection is equivalent to saying that the curve representing $x(u)$ in the ux-plane passes both the horizontal and vertical line tests.

For change of variables in multiple integrals, we have a function $f(x) : \mathbb{R}^n \to \mathbb{R}$ which we are integrating over a region D, and we wish to replace x with a new vector of variables $u \in \mathbb{R}^n$ via the change of variables $x = x(u)$. We suppose that E is the domain of $x(u)$ and that $x(u)$ as a function from E to D is a bijection. The inverse function theorem states that if we want to see whether $x(u)$ is a bijection, then we check that $\det\nabla x(u) \neq 0$ for all $u \in E$. If that is the case, then $E = \{u : x(u) \in D\}$ is the new region over which u is integrated. If we think of a function $x(u) : \mathbb{R} \to \mathbb{R}$, it makes sense that we should insist that dx/du is non-zero for $x(u)$ to be a bijection, since near a point where dx/du is zero, it is possible that the horizontal line test fails (as is the case for the example $x(u) = u^2$ given above).

It remains now to determine what the integrand becomes under the change of variables $x = x(u)$. The analog of the dx/du term above is precisely $|\det\nabla x(u)|$ – recall that $\nabla x(u)$

is the matrix whose ijth entry is $\partial x_i/\partial u_j$. This is the statement of the change of variables theorem:

Change of Variables Theorem.

> Let $D, E \subset \mathbb{R}^n$ and let $x = x(u)$ define a bijection $E \to D$ – in other words $\det\nabla x(u)$ is never zero for $u \in E$. Then for any function $f : D \to \mathbb{R}$,
>
> $$\int\int\cdots\int_D f(x)dV = \int\int\cdots\int_E f(x(u))|\det\nabla x(u)|dW$$
>
> where dW is the element of volume with respect to $u \in \mathbb{R}^n$.

In the case $n = 1$, we have $\det\nabla x(u) = \frac{dx}{du}$, which is exactly the term we get when applying the substitution rule to a single variable function.

The change of variables theorem can be justified by looking at the definition of multiple integrals in terms of grid partitions. Recall that

$$\int\int\cdots\int_D f(x)dV = \lim_{\|P\|\to 0}\sum_{I\in P} f(x_I)|I|.$$

We would now like to justify the change of variables formula. For simplicity, we assume that D is a box with partition P and we write

$$x(u) = (x_1(u), x_2(u), \dots, x_n(u)).$$

Let P be a grid partition of D and let x be a lower corner of a box $I \in P$. Then under the transformation $x = x(u)$, we get a partition of E into regions which are no longer necessarily boxes. The surfaces which cut E into pieces are given by equations $x_i(u) = a_i$ where a_i is a constant. Let Q be the partition of E into these pieces. For a region $J \in Q$ enclosed on all sides by surfaces $x_i(u) = a_i$ for $i = 1, 2, \dots, n$, we estimate the volume of J. Let x denote the lower left corner of J. If we pick any $n - 1$ of the surfaces bounding J, then their intersection is a curve. Consider one tangent vector to each of these curves starting from the point x. Then the volume of J is approximated by the volume of a parallelepiped whose sides are vectors in the directions of these tangent vectors. The tangent vectors themselves are given by the rows of the matrix $\nabla x(u)$ and the volume of the parallelepiped they define is therefore $|\det\nabla x(u)|$. Using these observations in the definition of the multiple integral, this justifies the appearance of the Jacobian determinant in the change of variables theorem.

We remark that the difficulty in making such changes of variables is to determine first what change of variables to make, and then to determine the new region E of integration.

18.2 Classical co-ordinate systems

Polar co-ordinates. For $(x, y) \neq (0, 0)$ the transformation $(x, y) \mapsto (r\cos\theta, r\sin\theta)$ where $r > 0$ and $0 \leq \theta < 2\pi$ is a bijection by the inverse function theorem, since

$$\nabla(r\cos\theta, r\sin\theta) = \begin{matrix} \cos\theta & -r\sin\theta \\ r\cos\theta & \sin\theta \end{matrix}$$

has determinant equal to r and $r > 0$. Therefore changing variables from (x, y) to (r, θ) is valid. By the above computation

$$\int\int_D f dV = \int\int_E f(r\cos\theta, r\sin\theta) \cdot r \cdot dW$$

where dW is the element of area in (r, θ)-co-ordinates. These are usually referred to as polar co-ordinates. It is very helpful to remember the identities $r^2 = x^2 + y^2$ and $\theta = \arctan(y/x)$ for $x \neq 0$ when evaluating integrals.

Example 1. Evaluate $\int\int_D y^3(x^2 + y^2)^{-3/2} dA$ when $D = \{(x, y) : x^2 + y^2 \leq 1, y \geq \frac{1}{2}\}$.

Solution. Changing to polar co-ordinates, we get

$$\int\int_E (r\sin\theta)^3 \cdot r^{-3} \cdot r dW.$$

The region E has to be computed. We observe that

$$E = \{(r, \theta) : \pi/6 \leq \theta \leq 5\pi/6, 1/(2\sin\theta) \leq r \leq 1\}$$

since the line $y = \frac{1}{2}$ is $r\sin\theta = 1/2$. So the integral is

$$\int_{\pi/6}^{5\pi/6} \int_{1/2\sin\theta}^{1} (r\sin\theta)^3 \cdot r^{-3} \cdot r dr d\theta = \int_{\pi/6}^{5\pi/6} \Big(\frac{1}{2}\sin^3\theta - \frac{1}{8}\sin\theta\Big) d\theta = \frac{\sqrt{3}}{4}.$$

The key here was to see what the region E looks like in polar co-ordinates. Deciding in which order to do the integration was not an issue, since the function $r\sin^3\theta$ is continuous on the region E.

Example 2. Determine $\int\int_D f dA$ where $D = \{(x, y) : 1 \leq x^2 + y^2 \leq 4\}$ and $f(x, y) = e^{-x^2-y^2}$ using the substitution $(x, y) = (r\cos\theta, r\sin\theta)$ where $r > 0$ and $0 \leq \theta < 2\pi$.

Solution. As above $\det\nabla(r\cos\theta), r\sin\theta) = r$. Now on D, we have that $1 \leq r \leq 4$. Therefore by the change of variables theorem,

$$\int\int_D f dA = \int\int_E e^{-r^2} r dW$$

where $E = \{(r,\theta) : 0 \leq \theta < 2\pi, 1 \leq r \leq 2\}$ and dW is the element of area in polar co-ordinates. Now re^{-r^2} is continuous so the integral on the right is an iterated integral: it is

$$\int_0^{2\pi}\int_1^2 re^{-r^2}drd\theta = \frac{1}{2}\int_0^{2\pi}\Big(\frac{1}{e} - \frac{1}{e}\Big)d\theta = \frac{\pi}{e}\Big(1 - \frac{1}{e}\Big).$$

This determines the integral. Note that the original integral in Cartesian co-ordinates could not be evaluated, since e^{-x^2} and e^{-y^2} have no explicit antiderivatives.

Spherical co-ordinates. In spherical co-ordinates, we define a point (x,y,z) in terms of co-ordinates (ρ,θ,ϕ) so that $\rho = \sqrt{x^2+y^2+z^2}$ is the distance from the point to the origin, θ is the angle between the x-axis and the line from the origin to the point $(x,y,0)$, and ϕ is the angle between the z-axis and line from the origin to (x,y,z). We insist that $0 < \phi < \pi$, $0 \leq \theta < 2\pi$, and $\rho > 0$. Note that in the xy-plane, this just reduces to polar co-ordinates, where $\phi = \pi/2$. The relationship between x,y,z and ρ,θ,ϕ can be found explicitly using some geometry:

$$\begin{aligned} x &= \rho\cos\theta\sin\phi \\ y &= \rho\sin\theta\sin\phi \\ z &= \rho\cos\phi \end{aligned}$$

The Jacobian determinant for this transformation is equal to $-\rho^2\sin\phi$. Since $0 < \phi < \pi$ and $\rho > 0$, we know that the function $(x,y,z) = (\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)$ is a bijection from $\mathbb{R}^3$ to $\mathbb{R}^3$, although technically we have to remove the entire z-axis since $0 < \phi < \pi$. It follows that

$$\int\int\int_D f dV = \int\int\int f(\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)\rho^2\sin\phi dW.$$

It is again handy to remember the identity $x^2+y^2+z^2 = \rho^2$.

Example 1. Evaluate $\int\int\int_D f dV$ where D is the unit ball and $f(x,y,z) = \exp(-(x^2 + y^2 + z^2)^{3/2})$.

Solution. Since $x^2+y^2+z^2 = \rho^2$ we get

$$\int\int\int_E e^{-\rho^3}\rho^2\sin\phi d\rho d\phi d\theta.$$

The order of integration is immaterial since all functions concerned are continuous. Now E is given by $\rho \leq 1$, so we get

$$\int_0^{2\pi}\int_0^{\pi}\int_0^1 e^{-\rho^3}\rho^2\sin\phi d\rho d\phi d\theta = \frac{4(e-1)\pi}{3}.$$