Lecture 19 - Wednesday May 13th

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Key words: Change of variables, Jacobian for implicit transformation, hyperbolic co-ordinates, rotation
Key concepts: Know how to deal with implicit transformations

19.1 Classical co-ordinate systems

Cylindrical co-ordinates. In cylindrical co-ordinates, a point (x, y, z) in $\mathbb{R}^3$ is represented by a triple (ρ, ϕ, z) where ρ denotes the distance to the projection of the point on the xy-plane to the origin, namely $\rho = \sqrt{x^2 + y^2}$, ϕ denotes the angle between the line from the origin to the point $(x, y, 0)$ and the x-axis, and z is the height of the point (x, y, z) above the xy-plane. Therefore with some geometry we obtain

$$\begin{aligned} x &= \rho \cos \phi \\ y &= \rho \sin \phi \\ z &= z \end{aligned}$$

as the equations representing the transformation from Cartesian to cylindrical co-ordinates. The Jacobian determinant for this transformation is easily seen to be ρ. We insist therefore that $\rho > 0$, and also $0 \leq \phi < 2\pi$.

Example 1. Find the volume of the region bounded above by the paraboloid $z = x^2 + y^2$, on the sides by the cylinder $x^2 + y^2 = 1$ and below by the xy-plane.

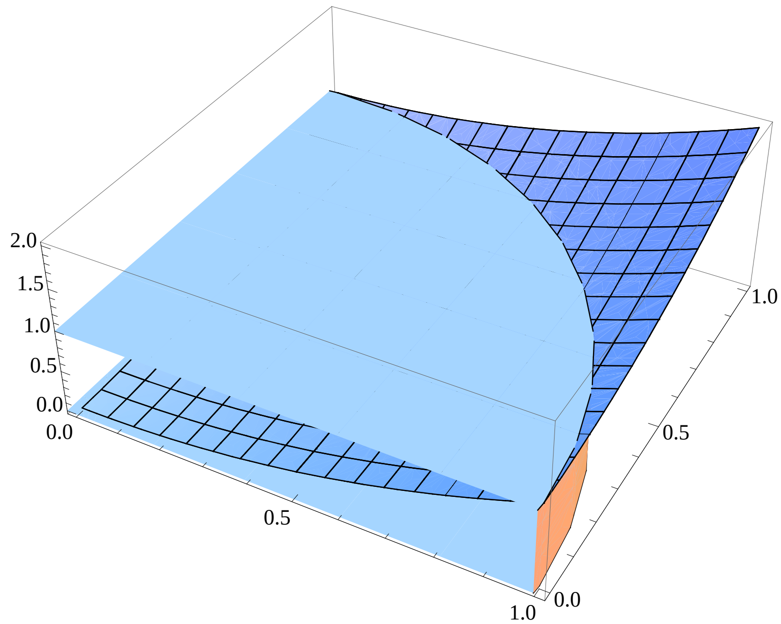
Solution. The volume is $\int \int \int_D 1 dV$ where D is the region in question. The region D can be represented easily in cylindrical co-ordinates as

$$D = \{(\rho, \phi, z) : 0 \leq \phi < 2\pi, 0 < \rho \leq 1, 0 \leq z \leq \rho^2\}$$

since $z \leq x^2 + y^2$ represents the region below the paraboloid. Therefore

$$\text{Volume } = \int_0^{2\pi} \int_0^1 \int_0^{\rho^2} \rho dz d\rho d\phi = \frac{\pi}{2}.$$

The region is drawn below.



The region $D = \{(x, y, z) : z \geq 0, z \leq x^2 + y^2, x^2 + y^2 \leq 1\}$

19.2 Further Transformations.

We have seen that in a triple integral whose integrand is of the form $f(x^2 + y^2 + z^2)$ or where the region of integration involves spherical pieces, it is suggested to use spherical co-ordinates. In the case where the integrand is of the form $f(x^2 + y^2, z)$ or the region of integration involves cylindrical pieces, the use of cylindrical co-ordinates is suggested. In a general situation, we may have to decide on a completely new co-ordinate system.

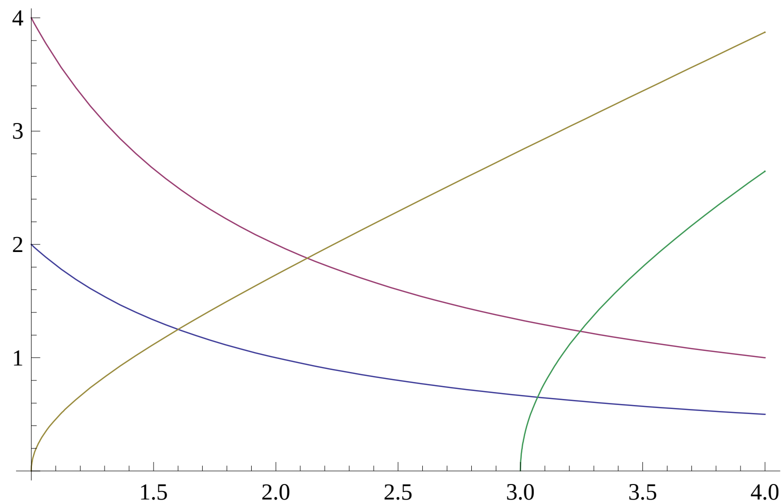
Also, we consider implicit transformations. So far we have considered transformations where $x = x(u)$, but sometimes it may be more natural to have an implicit or inverse transformation $u = u(x)$. For example, instead of being given $x = r\cos\theta$ and $y = r\sin\theta$, we might be given the same transformation in the form $r = \sqrt{x^2 + y^2}$ and $\theta = \arctan(y/x)$. To determine the Jacobian determinant $\det\nabla x(u)$, we use that $J = \nabla x(u)$ and $K = \nabla u(x)$ are inverse matrices, and which means $JK = I$, the identity matrix. By multiplicativity of determinants, this means $\det J \cdot \det K = 1$ and so we find

$$\det\nabla x(u) = \frac{1}{\det\nabla u(x)}.$$

In other words, when we are given the new co-ordinates u as a function $u(x)$ of the old co-ordinates, we just compute the Jacobian determinant $\det\nabla u(x)$, and then take the reciprocal to get $\det\nabla x(u)$. We give some examples below:

Example 1. (Hyperbolic co-ordinates) Evaluate $\int\int_D f dA$ where $f(x, y) = x^2 + y^2$ and D is the region in the plane bounded by $x^2 - y^2 = 1$, $x^2 - y^2 = 9$, $xy = 2$ and $xy = 4$.

Solution. While the function f suggests cylindrical co-ordinates, the region D does not. We will use hyperbolic co-ordinates $u = x^2 - y^2$ and $2xy = v$. The region of integration D is shown below:



The region D bounded by $xy = 2$, $xy = 4$, $x^2 - y^2 = 1$, $x^2 - y^2 = 9$

Under the transformation of co-ordinates, $D = \{(u, v) : 1 \leq u \leq 9, 4 \leq v \leq 8\}$. Notice also that

$$f(x, y)^2 = (x^2 + y^2)^2 = x^4 + y^4 + 2x^2y^2 = u^2 + v^2.$$

Therefore the integral is

$$\int_1^9 \int_4^8 \sqrt{u^2 + v^2} |\det(x(u, v), y(u, v))| dudv.$$

First we find the Jacobian of the transformation from (u, v) to (x, y): this is

$$J = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}.$$

Now

$$J = \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix}$$

so the determinant is $4(x^2 + y^2) = 4\sqrt{u^2 + v^2}$. Note that this is never zero on the region D. Now we use the fact

$$\det \nabla x(u) = \frac{1}{\det \nabla u(x)}.$$

So we have

$$\int \int_D f dA = \int_1^9 \int_4^8 \sqrt{u^2 + v^2} \cdot \frac{1}{|\det J|} dudv = \frac{1}{4} \int_1^9 \int_4^8 1 dudv = 8.$$

Example 2. (Rotation) Evaluate $\int\int_D f dA$ where $f(x,y) = \cos\left(\frac{x-y}{x+y}\right)$ and D is the region bounded by $x+y=1$, $x=0$ and $y=0$.

Solution. The region D is a triangle. We will let $x-y=u$ and $x+y=v$. This corresponds to rotating the axes through $\pi/4$ radians. The Jacobian determinant for this transformation is from (u,v) to (x,y) is

$$J = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

So the determinant $\det J = 2$. Now $x = (u+v)/2$ and so

$$D = \{(x,y) : 0 \le x \le 1, 0 \le x+y \le 1\} = \{(u,v) : 0 \le v \le 1, -v \le u \le v\}.$$

It follows that the integral is

$$\begin{aligned} \int\int_D f dA &= \frac{1}{2}\int_0^1 \int_{-v}^{v} \cos\left(\frac{u}{v}\right) du dv \\ &= \frac{1}{2}\int_0^1 v \sin \frac{u}{v}\Big]_{-v}^{v} dv \\ &= \sin 1 \int_0^1 v dv \\ &= \frac{1}{2}\sin 1. \end{aligned}$$

19.3 Applications.

Volume. One of the main applications of multiple integrals is determining volumes. If $D \subseteq \mathbb{R}^n$ is a bounded region in space, then the volume of D, when it exists, is

$$V = \int\int\cdots\int_D 1 dV.$$

This most pertinent in the case $n=3$, but is valid in all dimensions. It is a good exercise to verify that the volume of the ball $x^2+y^2+z^2 \le r^2$ in $\mathbb{R}^3$ is $4\pi r^3/3$, and even more interesting to determine the formula for the volume of a hypersphere $x_1^2+x_2^2+\cdots+x_n^2 \le r^2$.

Probability and Expectation. A function $f : \mathbb{R}^2 \to \mathbb{R}$ defines a probability distribution on $\mathbb{R}^n$ if $0 \le f(x) \le 1$ for all x and

$$\int\int\cdots\int_{\mathbb{R}^n} f dV = 1.$$

The expected value of a function $f : \mathbb{R}^n \to \mathbb{R}$ defined on $\mathbb{R}^n$ is the improper integral $\int\int\cdots\int f dV$ when it exists. The expected value or expectation is a concept from probability, but is heavily tied in to physics too. One of the most interesting derivations is the fact that $f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ defines a probability distribution on $\mathbb{R}$. Surprisingly, we will use multiple integrals and the change of variable theorem to see this.