Lecture 2 : Wednesday April 1st

jacques@ucsd.edu

Key concepts : Limit, continuity
Know how to find limits with ϵ-δ and show they don't exist and use properties of limits

2.1 Definition of limits

The definition of a limit for a function $f : \mathbb{R}^n \to \mathbb{R}$ of several variables is a natural extension of the limit for functions of one variable. The main difference is that instead of saying $|x - a| < \delta$, we say $d(x, a) < \delta$ where we recall that

$$d(x, a) = \sqrt{(x_1 - a_1)^2 + (x_2 - a_2)^2 + \cdots + (x_n - a_n)^2}.$$

In the case $n = 1$, notice that $d(x, a) = |x - a|$, so this agrees with the one-dimensional definition of the limit. So here is the definition of the limit upon which all our further calculus is based:

Definition of the limit
The limit of a function $f : \mathbb{R}^n \to \mathbb{R}$ as $x \to a$ is L

$$\lim_{x \to a} f(x) = L$$

if for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $d(x, a) < \delta$ then $|f(x) - L| < \epsilon$. The function f is continuous at a if $L = f(a)$.

This definition in words says that if x is close to a then $f(x)$ should be close to L. Limits of functions of several variables have many of the same properties as the limits we are familiar with for functions of one variable. These properties are listed below – here it is assumed that the limits of each of the functions f and g exist and g has a non-zero limit in the third property.

Properties of limits

- [Sum Rule] $\lim_{x \to a}(f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
- [Product Rule] $\lim_{x \to a}(f(x) \cdot g(x)) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$
- [Quotient Rule] $\lim_{x \to a}(f(x)/g(x)) = (\lim_{x \to a} f(x))/(\lim_{x \to a} g(x))$

An additional and useful property is the substitution rule for limits. If f is a one-variable function and $g : \mathbb{R}^n \to \mathbb{R}$, then if $\lim_{z\to L} f$ exists where $L = \lim_{x\to a} g(x)$, we have

Substitution rule

- $\lim_{x\to a} f(g(x)) = \lim_{z\to L} f(z)$.

A particularly simple case is where f and g are continuous everywhere; then the substitution rule just becomes $\lim_{x\to a} f(g(x)) = f(g(a))$.

2.2 Proving limits by definition

When using the ϵ-δ definition of the limit to prove $\lim_{x\to a} f(x) = L$, in practice we start by assuming $d(x, a) < \delta$ and then try to compute an ϵ such that $|f(x) - L| < \epsilon$, making sure that ϵ is a function of δ which tends to zero as δ tends to zero. Throughout this section p denotes a generic point (x, y).

Example 1. The function $f(x, y) = \sin(x^2 + y^2)$ we said we expect to be continuous everywhere. Let's check continuity at the origin – which we denote by 0 – via the definition. Since $f(0) = 0$, we want $\lim_{p\to 0} f(p) = 0$ for f to be continuous at 0. We will apply the definition of the limit with $a = 0$ and $L = 0$. By definition, we have to find a δ such that $|f(x, y) - L| < \epsilon$ whenever $d(p, a) < \delta$. Using $\sin\alpha \le \alpha$ for $\alpha > 0$, we get

$$\begin{aligned} |f(x,y) - L| < \epsilon \quad &\leftarrow \quad |\sin(x^2+y^2)| < \epsilon \\ &\leftarrow \quad |x^2+y^2| < \epsilon \\ &\leftarrow \quad \sqrt{x^2+y^2} < \sqrt{\epsilon} \quad \leftarrow \quad d(p,a) < \sqrt{\epsilon}. \end{aligned}$$

Therefore for every $\epsilon > 0$, there exists a $\delta > 0$ – namely $\delta = \sqrt{\epsilon}$ – such that if $d(p, a) < \delta$ then $|f(x, y) - L| < \epsilon$. So the limit is zero.

Note that we could have used the substitution rule with $f(z) = \sin z$ and $g(x, y) = x^2 + y^2$, since $\sin(x^2 + y^2) = f(g(x, y))$ in this case. Since $\lim_{p\to 0} g(p) = 0$ and $\lim_{z\to 0} f(z) = 0$ we get that $\lim_{p\to 0} \sin(x^2 + y^2) = 0$ without using the ϵ-δ definition.

Example 2. Show that $\lim_{(x,y)\to 0} f(x, y) = 0$ if $f(x, y) = x^2/\sqrt{x^2 + y^2}$.

Solution. We apply the definition of the limit with $a = 0$ and $L = 0$.

$$\begin{aligned} |f(x,y) - L| < \epsilon \quad &\leftarrow \quad \left|\frac{x^2}{\sqrt{x^2+y^2}}\right| < \epsilon \\ &\leftarrow \quad \frac{x^2+y^2}{\sqrt{x^2+y^2}} < \epsilon \\ &\leftarrow \quad \sqrt{x^2+y^2} < \epsilon \quad \leftarrow \quad d(p,a) < \epsilon. \end{aligned}$$

Taking $\delta = \epsilon$ in the definition of the limit, we conclude that $\lim_{(x,y)\to 0} f(x,y) = 0$.

Example 3. Show that $\lim_{(x,y)\to 0} h(x,y) = 1$ if $h(x,y) = \sin(x+y)/(x+y)$.

Solution. This is an example where we can use the properties of limits. We use the substitution rule. First observe that $g(x,y) = x+y$ is continuous so $\lim_{p\to 0} g(p) = 0$. Next we know that

$$\lim_{z\to 0} \frac{\sin z}{z} = 1$$

and since $h(x,y) = f(g(x,y))$ where $f(z) = \sin(z)/z$, we get from the substitution rule that

$$\lim_{p\to 0} h(x,y) = \lim_{z\to 0} f(z) = 1.$$

2.3 Showing a limit does not exist

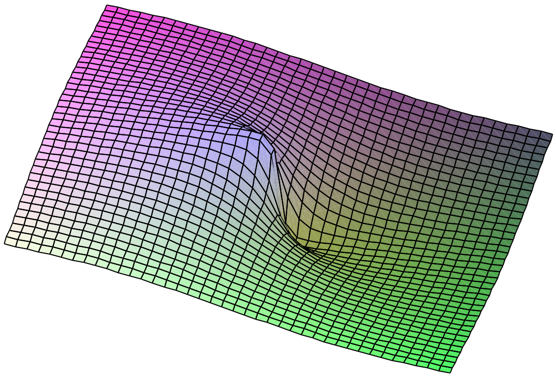
In the next examples we deal with limits that do not exist. For a one-variable function $f(x)$, one way to show the limit $\lim_{x\to a} f(x)$ does not exist is to prove that the left and right limits of $f(x)$ are different. For example, the function $f(x) = 0$ for $x < 0$ and $f(x) = 1$ for $x \geq 0$ is clearly discontinuous at the origin because the left and right limits are zero and one respectively. A similar thing often works for functions of several variables, except that we have to try limits along different directions instead of just left and right limits. Generally, for the limit of a function $f(x,y)$ as $(x,y) \to (a,b)$, we might let $y = mx$ where m is a real number and see whether the limits $\lim_{x\to a} f(x,mx)$ for each different value of m are equal. If they are not all equal, then the original limit does not exist. If they are equal, then we cannot conclude anything about the existence of the limit. Occasionally, it is even useful to consider instead of $y = mx$ some different paths, like $y = mx^2$ or $y = me^{-1/x^2}$, depending on the limit in question.

Example 4. Show that $\lim_{(x,y)\to 0} f(x,y)$ does not exist when $f(x,y) = x/\sqrt{x^2+y^2}$.

Solution. We put $y = mx$ so that

$$f(x,mx) = \frac{x}{\sqrt{x^2+m^2x^2}} = \frac{1}{\sqrt{1+m^2}}.$$

Now as $x \to 0$ this obviously depends on m – if $m = 0$ the limit is 1 whereas if $m = 1$ the limit is $1/\sqrt{2}$. Therefore the original limit does not exist.



$$f(x, y) = \frac{x}{\sqrt{x^2 + y^2}}$$

Note that the picture looks continuous, but this function is not continuous at the origin. As we said, if we walk along the line $y = 0$ we end up not agreeing with the walk along the the line $y = x$. Even more striking, if we let $x = 0$ and then take the limit as $y \to 0$ we get zero, whereas if we let $y = 0$ and take the limit as $x \to 0$ we get one.

In the next example, we see that we need to choose a different curve than $y = mx$ to show that the limit does not exist. The main point is that the curve $y = g(m, x)$ chosen must have the property that $\lim_{x\to 0} g(m, x) = 0$ for every value of m. This is a slightly tricky example, and is not examinable.

Example 5. Show that $\lim_{(x,y)\to 0} f(x, y)$ does not exist when $f(x, y) = x^2 \ln y^2$.

Solution. We try the curve $y = g(m, x) = e^{-1/mx^2}$ where $m > 0$. Notice this is valid since $\lim_{x\to 0} g(m, x) = 0$ for every value of $m > 0$. Then

$$\lim_{x\to 0} f(x, e^{-1/mx^2}) = \lim_{x\to 0} \frac{-2x^2}{mx^2} = \frac{-2}{m}.$$

Since this limit depends on m, the original limit does not exist.