Lecture 21 - Wednesday May 20th

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KEY WORDS: MOMENTS OF INERTIA, CENTER OF MASS, WORK

21.1 Moments of inertia and center of mass

Let W be a solid with density $\delta(x, y, z)$ at point (x, y, z). Then the moments of inertia of W with respect to the co-ordinate axes are

$$I_x = \iiint_W (y^2 + z^2)dV \quad I_y = \iiint_W (x^2 + z^2)dV \quad I_z = \iiint_W (x^2 + y^2)dV.$$

These quantities measure the difficulty with which an object is rotated about the axes. As an exercise, one can determine whether it is harder to rotate a sphere $x^2+y^2+z^2 = 1$ with uniform density $\delta(x) = 1$ around the z-axis or to rotate the cylinder $x^2 + y^2 \leq 1, z \leq 4/3$ with uniform density $\delta(x) = 1$ around the z-axis. Note that these objects clearly have the same mass. The center of mass of a solid object W in n dimensions is at the point $w = (w_1, w_2, \dots, w_n)$ defined by

$$w_i = \int\int \cdots \int_W \frac{x_i \delta(x)}{m} dV$$

where m is the mass of the object.

Example 1. Find the center of mass of $H = \{(x, y, z) : x^2 + y^2 + z^2 \leq 1, z \geq 0\}$ with uniform density.

Solution. We can assume that $\delta(x) = 1$ is the density function for the hemisphere, and the hemisphere is defined by $x^2 + y^2 + z^2 \leq 1$ where $z \geq 0$. Then the mass of the hemisphere is its volume, namely $\frac{2}{3}\pi$. By definition we have to determine

$$\frac{3}{2\pi} \iiint_H x dV \quad \frac{3}{2\pi} \iiint_H y dV \quad \frac{3}{2\pi} \iiint_H z dV.$$

It is easy to see that the first two integrals are zero (for example, convert to spherical co-ordinates). The third integral we convert to spherical co-ordinates to get

$$\frac{3}{2\pi} \int_0^{2\pi} \int_0^{\pi/2} \int_0^1 \rho^3 \sin\phi \cos\phi d\rho d\phi d\theta.$$

A quick calculation gives that this integral is

$$\frac{3}{4} \int_0^{\pi/2} \sin\phi \cos\phi d\phi = \frac{3}{8}.$$

So the center of mass is at $(0, 0, 3/8)$.

21.3 Work

If a particle moving in a straight line from a point p to a point q in space is subject to a constant force f, then the work done on the particle by the force f is defined to be $f \cdot (q-p)$, the scalar or dot product of f and the vector $q-p$. In words, the work done is the product of the displacement and the force exerted in the direction of displacement. Agreeing with the dot product, if a particle moves at right angles to a force f, then no work is done on it by that force.

In the case that the particle follows a particular curve γ from p to q, but the force f is still constant, we can imagine that the curve is partitioned into small pieces which will be approximated by straight lines (tangent line segments, in fact), and then add up the work done on each of these small pieces to approximate the work done. This leads us to a Riemann type sum, and the definition of work as an integral, even for non-constant force fields.

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be any vector field and suppose a particle moves in space along a curve γ subject to the force field f. Suppose that γ is parametrized by $r(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ for $a \leq t \leq b$. For example, the circle $x^2 + y^2 = 1$ could be parametrized by $r(t) = (\cos t, \sin t)$ in polar co-ordinates. Note there are many parametrizations possible for a given curve γ – the circle could also be parametrized as $(\sin t, \cos t)$ and so on. The work done on the particle moving along γ by the field f is defined by

$$\int_a^b f(r(t)) \cdot \frac{dr}{dt}\, dt$$

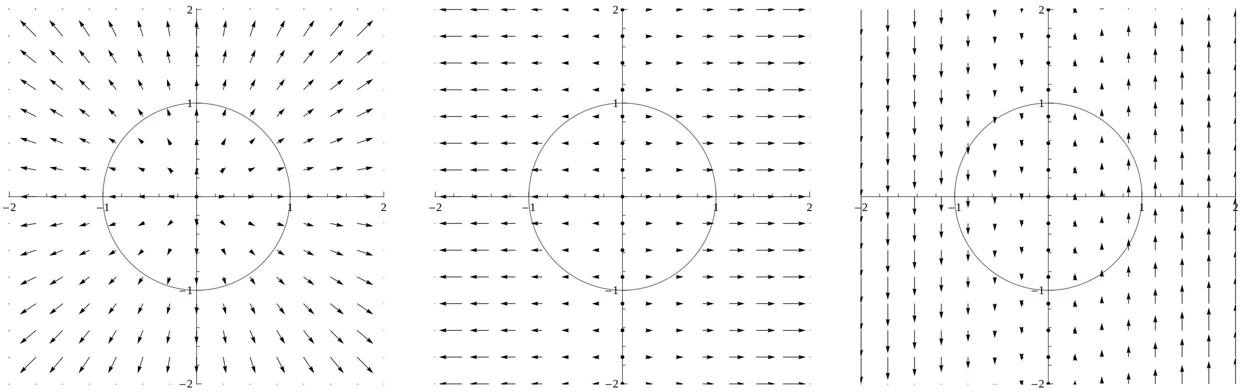
where again $f(r(t)) \cdot \frac{dr}{dt}$ is the scalar or dot product of $f(r(t))$ and $\frac{dr}{dt}$. We often refer to r as the position vector of the particle at time t. This integral is often written in shorthand as

$$\int_\gamma f \cdot dr$$

and is a special case of what will be called a line integral.

Example 1. Suppose that a particle moves along the circle $x = \cos\theta$ and $y = \sin\theta$ for $0 \leq \theta \leq 2\pi$. Consider the force fields $f(x,y) = (0,x)$, $f(x,y) = (x,0)$ and $f(x,y) = (x,y)$. Determine the work done by each of these force fields on the particle.

Solution. Intuitively, only the first force field does any work on a particle moving around the circle, since for $(x,0)$, the work done in moving the particle through one semicircle is cancelled out by the work done in moving the particle through the remaining semicircle, and for (x,y) the force field is perpendicular to the motion of the particle at every point around the circle.



Vector fields (x, y), $(x, 0)$ and $(0, x)$ in $\mathbb{R}^2$.

First note that $dr/dt = (dx/dt, dy/dt) = (-\sin\theta, \cos\theta)$ and so if $f(x, y) = (x, y)$ then

$$f(r(t)) \cdot \frac{dr}{dt} = f(\cos\theta, \sin\theta) \cdot (-\sin\theta, \cos\theta) = 0.$$

So clearly the work done is zero. For $f(x, y) = (x, 0)$, we get

$$f(r(t)) \cdot \frac{dr}{dt} = f(\cos\theta, \sin\theta) \cdot (-\sin\theta, \cos\theta) = -\sin\theta\cos\theta.$$

By definition, the work done is

$$-\int_0^{2\pi} \sin\theta\cos\theta d\theta = -\frac{1}{2}\int_0^{2\pi} \sin 2\theta d\theta = 0$$

as we expected. Finally, for $f(x, y) = (0, x)$ we get

$$f(r(t)) \cdot \frac{dr}{dt} = (0, \cos\theta) \cdot (-\sin\theta, \cos\theta) = \cos^2\theta.$$

By definition, the work done is

$$\int_0^{2\pi} \cos^2\theta d\theta = \pi.$$