Lecture 22 - Friday May 22nd

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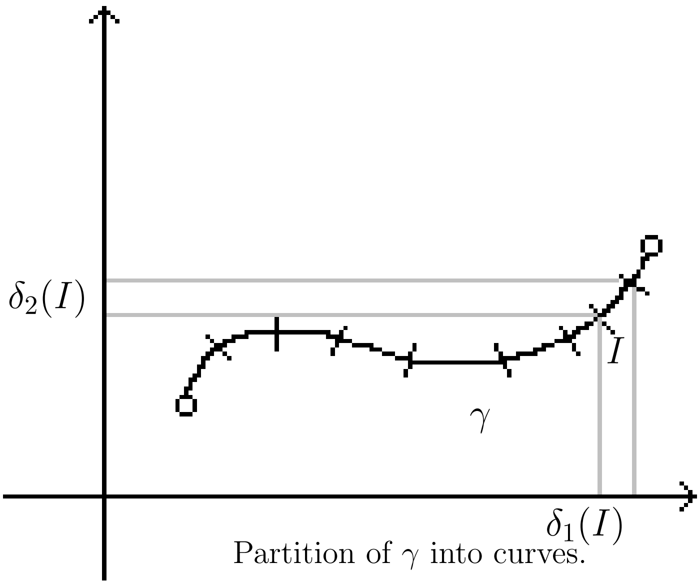
KEY WORDS: LINE INTEGRAL, CONSERVATIVE VECTOR FIELD, PARAMETRIZATION

22.1 Line Integrals

The Riemann integral $\int_a^b f(x)dx$ denotes the integral of the function f along the interval $[a, b]$. More generally, we could define integrals along curves instead of the straight line consisting of the interval $[a, b]$. These integrals are generally referred to as line integrals, and we will see via the major integral theorems of calculus how they relate to multiple integrals. Moreover they arise very naturally in applied mathematics and physics, for example, the notion of the work done by a force field on a particle moving through was seen to be a line integral in the last lecture. We begin the the definition of line integrals.

Let γ be a curve in $\mathbb{R}^n$ connecting points p and q. We assume $\gamma \in C^1$ – so γ has tangent lines at every point. Let $f(x) = (f_1(x), f_2(x), \ldots, f_n(x))$ be a function from $\mathbb{R}^n$ to $\mathbb{R}^n$ defined at all points of γ. Let P be a partition of γ into curves, and for a curve $I \in P$ let $\delta_j(I)$ denote the difference between the jth co-ordinate of the first point in I and the last point in I.



Partition of γ into curves.

Then we define the Riemann Sum

$$\sum_{j=1}^{n} \sum_{I \in P} f_j(x_I)\delta_j(I).$$

where x_I is an arbitrary point on the curve I. Let $\|P\|$ denote the largest of all the values of $\delta_j(I)$ for $I \in P$ and $j = 1, 2, \ldots, n$. If the limit

$$\lim_{\|P\|\to 0} \sum_{j=1}^{n} \sum_{I\in P} f_j(x_I)\delta_j(I)$$

exists, then the limit is called the line integral of f along γ and denoted by

$$\sum_{j=1}^{n} \int_{\gamma} f_j dx_j \quad \text{or} \quad \sum_{j=1}^{n} \int_{p}^{q} f_j dx_j.$$

If we let $r(x) = (x_1, x_2, \ldots, x_n)$ denote the position vector of x, and we define formally $dr = (dx_1, dx_2, \ldots, dx_n)$, then we may rewrite the line integral in the more succinct form

$$\int_{\gamma} f \cdot dr$$

where the dot denotes the scalar product of f and dr as vectors. If γ is the path taken by a particle moving through the force field f, then this line integral is exactly the definition of work given in §21.3. In fact, the evaluation of line integrals is done in a similar way, using parametrizations of γ. If γ is a curve in $\mathbb{R}^n$, then γ can be viewed as a function from some interval $[a, b]$ to $\mathbb{R}^n$. If $r(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ is such a function, then r is said to parametrize γ. For example, the unit circle γ centered at the origin can be parametrized by $r(t) = (\cos t, \sin t)$, and the straight line from $(0, 0, 0)$ to $(1, 1, 1)$ in $\mathbb{R}^3$ can be parametrized by $r(t) = (t, t, t)$. Note that parametrizations are not unique. They will play a major rôle in evaluating line integrals.

22.2 Evaluating line integrals

The notation $\int_\gamma f \cdot dr$ may seem a bit mysterious, but we can evaluate such a line integral by reducing it to an ordinary Riemann integral. The way in which we proceed depends on the way in which the curve γ is presented. We look in particular at the two-dimensional case.

A curve γ in the plane can sometimes be represented in the form $y = y(x)$ for $a \leq x \leq b$. This is one example amongst many of a parametrization of γ. In this case, given a vector field $f = (g, h)$, we have

$$\int_{\gamma} f \cdot dr = \int_{\gamma} g dx + h dy = \int_{a}^{b} \left[g(x, y(x)) + h(x, y(x)) \frac{dy}{dx} \right] dx.$$

In many cases, we cannot explicitly find a formula $y = y(x)$ for the curve γ. Nevertheless, if γ joins two points p and q, we can sometimes parametrize γ as $r(t) = (x(t), y(t))$ for $a \leq t \leq b$. Here $r(t)$ is sometimes referred to as the position vector of a particle moving along γ at time t. For instance, the circle $x^2 + y^2 = 1$ can be written as $(\cos t, \sin t)$ for $0 \le t \le 2\pi$. If we can write γ in this way, then

$$\int_\gamma f \cdot dr = \int_a^b f(x(t)) \cdot \frac{dr}{dt} dt$$

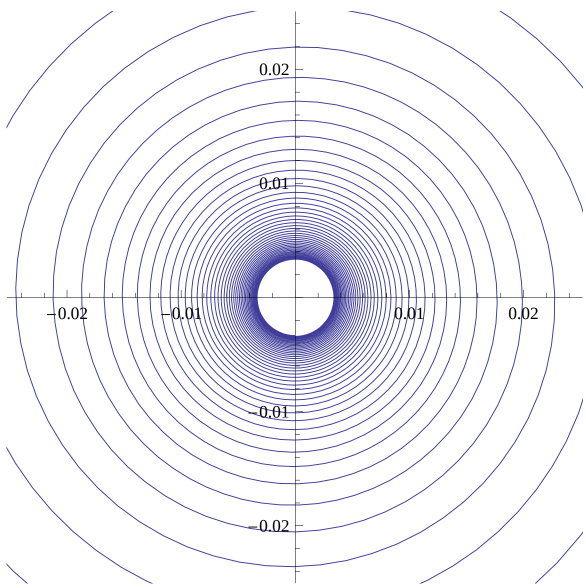
where $\frac{dr}{dt} = (\frac{dx}{dt}, \frac{dy}{dt})$. This is exactly the way we computed the work done by f on a particle moving along γ.

Example 1. Determine the line integral $\int_\gamma f \cdot dr$ along the curve γ defined by $y = x^2$ for $0 \le x \le 1$ where $f(x, y) = (xy, x + y)$.

Solution. As we said above, putting $y = x^2$, we get

$$\begin{aligned} \int_\gamma f \cdot dr &= \int_\gamma xy dx + (x + y) dy \\ &= \int_0^1 [x^3 + (x + x^2)(2x)] dx \\ &= \frac{17}{12}. \end{aligned}$$

Example 2. Determine the line integral $\int_\gamma f \cdot dr$ along the curve γ defined in polar co-ordinates by $r = 1/\theta$ for $\theta \ge \pi/2$ and where $f(x, y) = (x, y)$ for $x \neq 0$ and $y \neq 0$. In this case the curve γ is an infinite spiral into the origin starting at $(0, 1)$.



Infinite spiral γ.

Solution. We are given the parametrization of γ: it is $(r\cos t, r\sin t) = (t^{-1}\cos t, t^{-1}\sin t)$ in polar co-ordinates for $t > 0$. Then

$$f(x,y) = f(t^{-1}\cos t, t^{-1}\sin t) = (t^{-1}\cos t, t^{-1}\sin t).$$

Now if $r(t) = (t^{-1}\cos t, t^{-1}\sin t)$ then

$$\frac{dr}{dt} = \Big(\frac{-\sin t}{t} - \frac{\cos t}{t^2}, \frac{\cos t}{t} - \frac{\sin t}{t^2}\Big).$$

Therefore

$$f(r(t)) \cdot \frac{dr}{dt} = -\frac{\cos^2 t}{t^3} - \frac{\sin^2 t}{t^3} = -\frac{1}{t^3}.$$

So the line integral is

$$\int_\gamma f \cdot dr = \int_{\pi/2}^{\infty} \frac{1}{t^3} dt = \frac{2}{\pi^2}.$$

If we interpret this as work done, we are saying that a finite amount of work has to be done by f to move a particle along the spiral.

Example 3. Determine the work done by the force field $f(x,y) = (\sin x, \sin y)$ when a particle is moved counterclockwise around the boundary of the unit square $[0,\pi] \times [0,\pi]$ starting at $(\pi, 0)$.

Solution. By definition, from §21.3 the work done is

$$\int_\gamma f \cdot dr.$$

It is not hard to see that line integrals are linear, so if γ consists of curves $\gamma_1, \gamma_2, \dots, \gamma_k$ end to end, then

$$\int_\gamma f \cdot dr = \sum_{i=1}^{k} \int_{\gamma_i} f \cdot dr$$

provided all the line integrals exist. In our current situation, we have that γ, the boundary of the unit square, consists of the four straight line segments $x = \pi$, $y = \pi$, $x = 0$ and $y = 0$. Therefore

$$\int_\gamma f \cdot dr = \int_0^\pi f(\pi,y)\cdot(dx,dy) + \int_\pi^0 f(x,\pi)\cdot(dx,dy) + \int_\pi^0 f(0,y)\cdot(dx,dy) + \int_0^\pi f(x,0)\cdot(dx,dy).$$

It is important to note that since we are moving counterclockwise around the square, the limits of integration for the line integral along the lines $y = \pi$ and $x = 0$ are from π to zero. Now we evaluate each of these integrals: the first integral is

$$\int_0^\pi f(\pi,y) \cdot (dx,dy) = \int_0^\pi 0dx + \sin y dy = \int_0^\pi \sin y dy = 2.$$

The second integral is

$$\int_{\pi}^{0} f(x,\pi)\cdot(dx,dy) = \int_{\pi}^{0} \sin x = -2.$$

The third integral is

$$\int_{\pi}^{0} f(0,y)\cdot(dx,dy) = \int_{\pi}^{0} \sin y dy = -2.$$

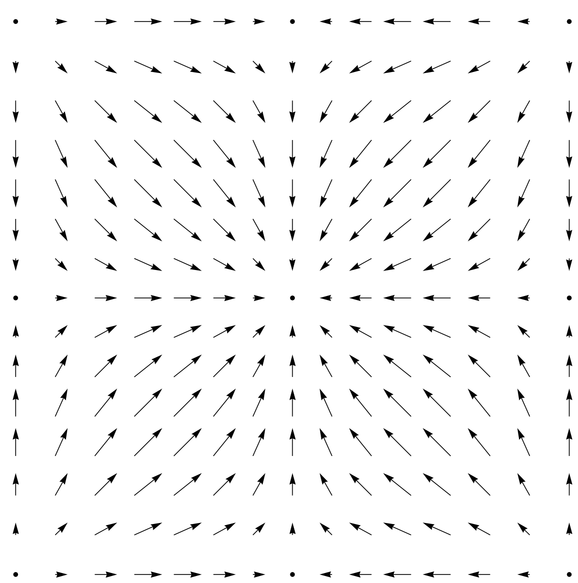
The final integral is

$$\int_{0}^{\pi} f(x,0)\cdot(dx,dy) = \int_{0}^{\pi} \sin x dx = 2.$$

Therefore

$$\int_{\gamma} f\cdot dr = 2 + (-2) + (-2) + 2 = 0,$$

and so no work is done on the particle. This should be clear once the vector field is plotted:



Vector field $(\sin x, \sin y)$.

Example 4. Evaluate $\int_{\gamma}(yzdx + xzdy + xydz)$ where γ is the straight line from $(0,0,0)$ to $(1,1,1)$. What if instead we consider the parabolic curve (x,x,x^2) from $(0,0,0)$ to $(1,1,1)$? What if we consider the curve $((e^x-1)/(e-1),(1-e^{-x})/(1-e^{-1}),x)$?

Solution. We can parametrize the straight line as (t,t,t) for $0 \le t \le 1$. Then the line integral is

$$\int_{0}^{1} t^2dt + t^2dt + t^2dt = 1.$$

For the parabolic curve we actually get the same answer:

$$\int_0^1 t^3 dt + t^3 dt + 2t^3 dt = 1.$$

For the last curve, we get

$$\int_0^1 \frac{te^t(1-e^{-t})}{(e-1)(1-e^{-1})}dt + \frac{te^{-t}(e^t-1)}{(e-1)(1-e^{-1})}dt + \frac{(1-e^{-t})(e^t-1)}{(1-e^{-1})(e-1)}dt = 1.$$

We should be asking at this stage why for these three very different curves we keep getting the same answer. We will see later that this is because the vector field (yz, xz, xy) is a conservative vector field – the line integral will be independent of the path taken.

22.3 Change of variables in line integrals

A curve γ in space generally has many parametrizations. How do two different parametrizations $r_1(t)$ and $r_2(t)$ affect the line integral of a vector field on γ? This depends on whether the parametrizations maintain the orientation of γ. Suppose we are given γ parametrized by $r_1(t)$ for $a \leq t \leq b$. Then we perceive γ as being oriented from $r_1(a)$ to $r_1(b)$. For example, the unit circle parametrized as $(\cos t, \sin t)$ for $0 \leq t \leq 2\pi$ is oriented counterclockwise, whereas the unit circle $(\cos t, -\sin t)$ is oriented clockwise. The main point is that the only difference in reparametrization can be the orientation of the curve, which only affects the sign of the integral.

Proposition.

Let γ be a curve parametrized by $r_1(t)$ and let δ be parametrized by $r_2(t)$ for $a \leq t \leq b$. If the line integral exists, and δ is the same curve as γ with the same orientation as γ, then

$$\int_\gamma f \cdot dr = \int_\delta f \cdot dr$$

for any vector field f which is C^1 on γ. If δ has the opposite orientation to γ, then

$$\int_\gamma f \cdot dr = -\int_\delta f \cdot dr.$$

So the situation could not be simpler: all we have to know is the effect on the orientation of γ when reparametrizing γ. A word of caution: if $r(t)$ is a parametrization of a curve γ, then when we reparametrize γ we must make sure that every point in γ is visited the same number of times as it was in the parametrization $r(t)$. For example, $(\cos t, \sin t)$ for

$0 \le t \le 2\pi$ and $(\cos t, \sin t)$ for $0 \le t \le 4\pi$ both represent unit circles, but in the second parametrization the circle is traversed twice. This may lead to different answers when computing line integrals.

Example 1. We parametrize the unit circle as $r(t) = (\cos \alpha t, \sin \alpha t)$ for $0 \le t \le 2\pi$, where α is a positive integer. We wish to compute the line integral

$$\int_\gamma f \cdot dr$$

where γ is the unit circle parametrized above and $f(x, y) = (x - y, x + y)$.

Solution. We are given the parametrization so

$$\int_\gamma f \cdot dr = \int_0^{2\pi} (\cos t - \sin t, \cos t + \sin t) \cdot (\cos \alpha t, \sin \alpha t) dt = \int_0^{2\pi} \cos((\alpha - 1)t) + \sin((\alpha - 1)t) dt$$

using the formulas $\cos(a-b) = \cos a \cos b + \sin a \sin b$ and $\sin(a-b) = \sin a \cos b - \sin b \cos a$. If $\alpha = 1$ this integral is 2π. If $\alpha > 1$ then the integral is zero. Note that α is the number of times we go around the unit circle, and the line integrals depends on α.

22.4 Conservative vector fields

For the vector field $f(x, y, z) = (yz, xz, yz)$, we saw that the line integrals along three very different curves from $(0, 0, 0)$ to $(1, 1, 1)$ were all the same. This is not just a coincidence, but rather a special property of vector fields. A vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is called conservative if there exists a function $F : \mathbb{R}^n \to \mathbb{R}$ such that $f = \nabla F$. For example, the field we just mentioned is conservative: by simple obervation, if $F(x, y, z) = xyz$, then $\nabla F = (yz, xz, yz)$. And it is this property of vector fields that is responsible for the line integral along all paths between two points being the same.

Theorem.

Let $F : \mathbb{R}^n \to \mathbb{R}$ be of class C^1 and let γ be a C^1 curve in $\mathbb{R}^n$ from point p to point q. Then

$$\int_\gamma \nabla F \cdot dr = F(q) - F(p).$$

This theorem is saying that if f is conservative, and we can determine F such that $f = \nabla F$, then the line integral of f does not depend on the curve γ between two points, but only its endpoints.

Example 1. We saw $f(x, y, z) = (yz, xz, yz)$ satisfies $f = \nabla F$ where $F = xyz$. By the theorem, if γ is any C^1 curve joining $(0, 0, 0)$ to $(1, 1, 1)$, then

$$\int_\gamma f \cdot dr = \int_\gamma \nabla F \cdot dr = F(1, 1, 1) - F(0, 0, 0) = 1.$$

This agrees with our preceding answer. Also note that the vector field $f(x, y) = (\sin x, \sin y)$ is conservative since $f = \nabla F$ where $F(x, y) = (-\cos x, -\cos y)$ by inspection. This explains why the integral of f around the square $[0, 1] \times [0, 1]$ is zero as we saw before.

The reason the theorem is true is that if we parametrize γ by $r(t)$ for $a \leq t \leq b$, then by definition

$$\int_\gamma \nabla F \cdot dr = \int_a^b \nabla F(r(t)) \cdot \frac{dr}{dt} dt = \int_a^b \frac{dF}{dt} dt$$

using the chain rule. By the fundamental theorem of calculus this is just $F(r(b)) - F(r(a))$ which is precisely $F(q) - F(p)$.