Lecture 23 - Wednesday May 27th

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KEY WORDS: CURL, CONSERVATIVE VECTOR FIELD, FORMAL DETERMINANT

23.1 Conservative vector fields

Recall that $f : \mathbb{R}^n \to \mathbb{R}^n$ is conservative if $f = \nabla F$ for some $F : \mathbb{R}^n \to \mathbb{R}$. A test for conservativity of a given vector field in $\mathbb{R}^2$ or $\mathbb{R}^3$ is done using the curl of the vector field.

Definition.

> The curl of a vector field (f, g, h) in $\mathbb{R}^3$, denoted curl(f) or $\nabla \times f$, is the vector $(h_y - g_z, f_z - h_x, g_x - f_y)$.

An easy way to remember the formula for the curl is to write it as a formal determinant:

$$\nabla \times f = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}.$$

Here we use the notation $i = (1,0,0)$, $j = (0,1,0)$ and $k = (0,0,1)$, and when we multiply $\frac{\partial}{\partial x}$ by g for example, we formally obtain g_x. As an exercise, verify that this gives the above formula from the definition. We now come to a simple test for a vector field $f : \mathbb{R}^3 \to \mathbb{R}^3$ to be conservative. An oriented curve γ in $\mathbb{R}^3$ is simple if it the image of a one-to-one piecewise C^1 map $r : [a,b] \to \mathbb{R}^3$. It is closed if it is one-to-one on $(a,b]$ and $r(a) = r(b)$.

Test for Conservativity.

1. A vector field f is conservative if and only if $\nabla \times f = 0$.
2. A vector field f is conservative if and only if for any two simple oriented curves γ and δ with the same endpoints, $\int_\gamma f \cdot dr = \int_\delta f \cdot dr$.
3. A vector field f is conservative if and only if for any simple oriented closed curve γ, $\int_\gamma f \cdot dr = 0$.

Example 1. Let $f(x,y,z) = (yz, xz, xy)$. We saw that this is conservative since $f = \nabla F$ where $F = xyz$. Now we check it using the test

$$\begin{aligned} \nabla \times f &= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix} \\ &= i\Big(\frac{\partial xy}{\partial y} - \frac{\partial xz}{\partial z}\Big) - j\Big(\frac{\partial xy}{\partial x} - \frac{\partial yz}{\partial z}\Big) + k\Big(\frac{\partial xz}{\partial x} - \frac{\partial yz}{\partial y}\Big) \\ &= (x - x, y - y, z - z) = 0. \end{aligned}$$

Therefore f is conservative.

Example 2. Determine $\int_\gamma f \cdot dr$ when γ is the curve $(t^t, \cos \pi t^2, \log t)$ for $1 \leq t \leq 2$ and $f(x, y, z) = (1 + y + yz, x + xz, xy)$.

Solution. The line integral will be impossible to evaluate explicitly unless we can show f is conservative. Now it is straightforward to check that $\nabla \times f = 0$ so by the test for conservativity, f is conservative. This means

$$\int_\gamma f \cdot dr = F(4, 1, \log 2) - F(1, -1, 0)$$

if $f = \nabla F$. There are two ways to proceed. One is to try to find F. The other is to use part 2 of the test to replace γ by a much simpler curve joining $(1, -1, 0)$ to $(4, 1, \log 2)$. For example, we could replace γ by $r(t) = (3t + 1, 2t - 1, t \log 2)$ for $0 \leq t \leq 1$. Then $\frac{dr}{dt} = (3, 2, \log 2)$ and the line integral is

$$\begin{aligned} \int_\gamma f \cdot dr &= \int_0^1 f(r(t)) \cdot \frac{dr}{dt} dt \\ &= \int_0^1 \Big[3[2t + (2t-1)(t \log 2)] + 2[(3t+1)(1 + t\log 2)] + \log 2(3t+1)(2t-1)\Big]\, dt \\ &= 8 + 4\log 2. \end{aligned}$$

It is important that this replacement of γ is only possible because f is conservative (by part 2 of the test). Now let's use the alternative, which is to compute F such that $f = \nabla F$. We need

$$\frac{\partial F}{\partial x} = 1 + y + yz \quad \frac{\partial F}{\partial y} = x + xz \quad \frac{\partial F}{\partial z} = xy.$$

Such equations for unknown F are called partial differential equations, and generally very difficult to solve. In this situation, however, we can solve them as follows. Since $F_x = 1 + y + yz$, we know

$$F(x, y, z) = x + xy + xyz + c(y, z)$$

because when we take derivatives in x the function $c(y, z)$ disappears and we get $1+y+yz$. This is like integrating in the usual sense, except that the arbitrary constant of integration is now a function of y and z. Since $F_y = x + xz$ we also have

$$F(x, y, z) = xy + xyz + d(x, z)$$

for some function $d(x, z)$. Finally $F_z = xy$ so

$$F(x, y, z) = xyz + e(x, y)$$

for some function $e(x, y)$. We have concluded

$$F(x, y, z) = x + xy + xyz + c(y, z) = xy + xyz + d(x, z) = xyz + e(x, y)$$

from which we get $c(y,z)=0$, $d(x,z)=x$ and $e(x,y)=x+xy$. Therefore

$$F(x,y,z)=x+xy+xyz.$$

Now the integral is

$$\int_\gamma f\cdot dr = F(4,1,\log 2)-F(1,-1,0)=8+4\log 2.$$

23.2 Line Integrals of Scalar Fields

A vector field is a function $f:\mathbb{R}^n\to\mathbb{R}^n$. A scalar field is a function $f:\mathbb{R}^n\to\mathbb{R}$. We have already defined line integrals of vector fields, and we will now define line integrals for scalar fields. If γ is a curve in $\mathbb{R}^n$ parametrized by a function $r(t):[a,b]\to\mathbb{R}^n$, then the line integral of a scalar field f along γ is defined by

$$\int_\gamma f ds = \int_a^b f(r(t))\Big\|\frac{dr}{dt}\Big\|dt$$

where

$$\Big\|\frac{dr}{dt}\Big\| = \sqrt{\Big(\frac{dx_1}{dt}\Big)^2+\Big(\frac{dx_2}{dt}\Big)^2+\cdots+\Big(\frac{dx_n}{dt}\Big)^2}.$$

The notation ds refers to the element of arclength. When $f=1$, then the line integral of f along γ is just the length of the curve γ.

Example 1. Determine the line integral of the scalar field $f(x,y)=x^2+y^2$ over the spiral $r=\theta$ for $0\le\theta\le 2\pi$ in polar co-ordinates.

Solution. The curve γ given by $r=\theta$ is parametrized by

$$r(t)=(x(t),y(t))=(r\cos t, r\sin t)=(t\cos t, t\sin t).$$

So

$$\frac{dr}{dt}=(\cos t - t\sin t, \sin t + t\cos t).$$

Therefore

$$\begin{aligned}\int_\gamma f ds &= \int_0^{2\pi} f(t\cos t, t\sin t)\sqrt{\cos^2 t + t^2\sin^2 t - 2t\cos t\sin t + \sin^2 t + t^2\cos^2 t + 2t\sin t\cos t}dt\\ &= \int_0^{2\pi} t^2\sqrt{1+t^2}dt\\ &=\end{aligned}$$

Example 2. Show that the length of $y=f(x)$ for $a\le x\le b$ is

$$\int_a^b\sqrt{1+\Big(\frac{df}{dx}\Big)^2}dx.$$

Solution. The line integral representing length of a curve γ given by $(x(t), y(t))$ is

$$\int_\gamma 1\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}dt.$$

Now we parametrize $y = f(x)$ using x as a parameter: so $r(x) = (x, f(x))$ is the parametrization. Putting this in the formula, we get the result. Now for example we could find the length of the parabola $y = \frac{1}{2}x^2$ for $-1 \leq x \leq 1$: it is

$$\int_{-1}^{1} \sqrt{1+x^2}dx = \sqrt{2} + \operatorname{arcsinh}(1).$$

23.3 Surface Integrals

Just as we have generalized Riemann integrals $\int_a^b f(x)dx$ to line integrals $\int_\gamma f \cdot dr$, we can generalize double integrals $\iint_D f dA$ to integrals over surfaces, which we will refer to as surface integrals. We begin with surface integrals of scalar fields. A surface Σ in $\mathbb{R}^3$ can be written in parametric form using two parameters. Thus $s(u,v) = (x(u,v), y(u,v), z(u,v))$ would be an example of a parametrized surface in $\mathbb{R}^3$, where u, v are in some set $D \subseteq \mathbb{R}^2$. For instance, $s(u,v) = (\cos u \sin v, \sin u \sin v, \cos v)$ parametrizes the unit sphere in $\mathbb{R}^3$ if $0 \leq u < 2\pi$ and $0 < v < \pi$.

Given a parametrization $s(u,v)$ of a given surface, we can define tangent vectors to the surface. The tangent vectors are

$$\begin{aligned} T_u &= \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) \\ T_v &= \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right). \end{aligned}$$

These are really important in the definition of surface integrals.

DarkGreen**Definition.**

> Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a scalar field defined on a surface S parametrized by $s(u,v) = (x(u,v), y(u,v), z(u,v))$ for $(u,v) \in D$ with tangent vectors T_u and T_v. Then the surface integral of f over S is defined by
>
> $$\iint_S f dS = \iint_D f(s(u,v)) \|T_u \times T_v\| du dv$$
>
> when this integral exists.

Recall that $T_u \times T_v$ is the vector or cross product of T_u and T_v, determined from the formal determinant

$$\begin{vmatrix} i & j & k \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}.$$

So this is the vector $(y_u z_v - z_u y_v, z_u x_v - x_u z_v, x_u y_v - y_u x_v)$. After computing this, we then take the norm, $\|T_u \times T_v\|$, to get

$$\|T_u \times T_v\| = \sqrt{(y_u z_v - z_u y_v)^2 + (z_u x_v - x_u z_v)^2 + (x_u y_v - y_u x_v)^2}.$$

In the case that $f = 1$, the integral above represents the surface area of the surface S.

Example 1. Suppose that S is given by $z = f(x, y)$ for $(x, y) \in D$. Find a formula for the surface area of S.

Solution. We parametrize the surface as $s(x, y) = (x, y, f(x, y))$ for $(x, y) \in D$. The tangent vectors are $T_x = (1, 0, f_x)$ and $T_y = (0, 1, f_y)$ and so we check that

$$\|T_x \times T_y\| = \sqrt{1 + f_x^2 + f_y^2}.$$

It follows that the surface area is

$$\iint_S 1 dS = \iint_D \sqrt{1 + f_x^2 + f_y^2} dxdy.$$

Example 2. Evaluate $\iint_S z^2 dS$ where S is the surface of the cube $[0, 1] \times [0, 1] \times [0, 1]$.

Solution. The surface of the cube consists of six plane pieces. We add up the surface integrals over each of the plane pieces. For the topmost plane piece, we have $z = 1$ and $x, y \in [0, 1] \times [0, 1]$. So we parametrize this surface as $(x, y, 1)$ for $(x, y) \in [0, 1] \times [0, 1]$. Then $T_x = (1, 0, 0)$ and $T_y = (0, 1, 0)$ so $\|T_x \times T_y\| = 1$. Thus the integral over this piece is

$$\int_0^1 \int_0^1 1 dxdy = 1.$$