Lecture 24 - Friday May 29th

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KEY WORDS: SURFACE INTEGRALS OF VECTOR FIELDS

24.1 Surface integrals of vector fields

We now come to surface integrals of vector fields. The motivation for line integrals of vector fields was the physical notion of work; here the motivation is flux. If a vector field f denotes the velocity vector field of fluid flowing through a surface, then the flux is the amount of fluid flowing through the surface per unit time. The flux will precisely be defined by the surface integral of the vector field.

Definition.

> Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field defined on a surface S parametrized by $R(u,v) = (x(u,v), y(u,v), z(u,v))$ for $(u,v) \in D$ with tangent vectors T_u and T_v. Then the surface integral of f over S is defined by
>
> $$\iint_S f \cdot dR = \iint_D f(R(u,v)) \cdot (T_u \times T_v) dudv$$
>
> when this integral exists.

The flux through S for a fluid with velocity field f is $\iint_S f \cdot dS$.

Example 1. Compute the flux across the unit hemisphere if $w(x,y,z) = (0,0,z)$ denotes the velocity vector field of fluid flowing through the hemisphere.

Solution. We parametrize the hemisphere as $R(x,y) = (x, y, \sqrt{1-x^2-y^2})$ for $x^2+y^2 \leq 1$. Then

$$T_x = \Big(1, 0, \frac{-x}{\sqrt{1-x^2-y^2}}\Big) \quad T_y = \Big(0, 1, \frac{-y}{\sqrt{1-x^2-y^2}}\Big).$$

Now

$$T_x \times T_y = \Big(\frac{x}{\sqrt{1-x^2-y^2}}, \frac{y}{\sqrt{1-x^2-y^2}}, 1\Big)$$

and therefore

$$\begin{aligned} \iint_S w \cdot dR &= \iint_{x^2+y^2 \leq 1} (0,0,\sqrt{1-x^2-y^2}) \cdot \Big(\frac{x}{\sqrt{1-x^2-y^2}}, \frac{y}{\sqrt{1-x^2-y^2}}, 1\Big) dxdy \\ &= \iint_{x^2+y^2 \leq 1} \sqrt{1-x^2-y^2} dxdy. \end{aligned}$$

Changing to polar co-ordinates we get

$$\int_0^{2\pi} \int_0^1 \sqrt{1-r^2} r dr d\theta = \frac{2\pi}{3}.$$

24.2 Parametrizations and orientations

For line integrals, the sign of the integral depends on the orientation of the curve we are integrating over. There is a similar notion of orientation for surfaces. An oriented surface us a two-sided surface with one sided prescribed as the positive side and the other side as the negative side. At each point on the positive side there is a normal vector which is the negative of the normal vector at the same point but on the negative side. If S is such a surface parametrized by $R(u,v)$, let $n(P)$ denote the unit normal to S at a point P. Since $(T_u \times T_v)/\|T_u \times T_v\|$ is also a unit normal to S, and S is two-sided, we have at the point $P = s(u,v)$,

$$\frac{T_u \times T_v}{\|T_u \times T_v\|} = \pm n(P).$$

We say that $R(u,v)$ is orientation-preserving if $n(P)$ and the vector on the left are equal, and orientation-reversing otherwise.

Example 2. The unit sphere is oriented by considering the outward normal vector $n(x,y,z) = (x,y,z)$ at the point (x,y,z) on the sphere. Parametrize the unit sphere by $R(u,v) = (\cos u \sin v, \sin u \sin v, \cos v)$ for $0 \leq u < 2\pi$, $0 < v < \pi$. Then

$$T_u = (-\sin u \sin v, \cos u \sin v, 0) \quad T_v = (\cos u \cos v, \sin u \sin v, -\sin v).$$

If n denotes the normal vector at a point $R(u,v)$, then a computation gives

$$\frac{T_u \times T_v}{\|T_u \times T_v\|} = -n.$$

Therefore the given parametrization is orientation reversing. We can make it orientation preserving by swapping u and v.

Proposition.

> Let S be an oriented surface and f a continuous vector field defined on S. If $R(u,v)$ and $S(u,v)$ are orientation preserving parametrizations of S, then the surface integrals of f over S with respect to each parametrization are equal. If $R(u,v)$ is orientation preserving and $S(u,v)$ is orientation reversing, then the integrals differ by a factor -1.

In fact, the surface integral of a vector field can be reduced to a surface integral of a scalar field as follows: if R is an orientation preserving parametrization of S, and $n = (T_u \times T_v)/\|T_u \times T_v\|$, then

$$\iint_S f \cdot dR = \iint_D f \cdot (T_u \times T_v) dudv = \iint_D (f \cdot n)\|T_u \times T_v\| dudv$$

and then we recall that $dS = \|T_u \times T_v\| du dv$ to get

$$\iint_S f \cdot dR = \iint_S (f \cdot n) dS.$$

This is saying that the surface integral of the vector field f is the same as the surface integral of the scalar field consisting of the normal component of f over S. Recalling the interpretation of the surface integral of f as the flux through S, this makes sense. It can also help to compute a surface integral as the integral of $f \cdot n$ rather than the original integral.

Example 3. Evaluate $\iint_S f \cdot dR$ when $f(x, y, z) = (x, y, z)$ and S is the unit sphere oriented outward.

Solution. We jump immediately to the integral of $f \cdot n$ where $n = (x, y, z)/\sqrt{x^2 + y^2 + z^2}$ is the outward unit normal to the sphere at (x, y, z). Then $f \cdot n = \sqrt{x^2 + y^2 + z^2}$. Since this is identically one on S, we have

$$\iint_S f \cdot dR = \iint_S f \cdot n dS = \iint_S 1 dS = 4\pi$$

which is just the surface area of S. This was much simpler than computing the original integral.

24.3 The four types of integrals

We have encountered four types of integrals: line integrals of vector and scalar fields, and surface integrals of vector and scalar fields. In summary, they are computed as follows, using the notation $r(t), R(u, v), T_u$ and T_v that we have been using all along: if γ is a curve and Σ is a surface then

$$\begin{aligned}
\int_\gamma f \cdot dr &= \int_a^b f(r(t)) \cdot \frac{dr}{dt}\, dt \quad \text{[Work integral]} \\
\int_\gamma f ds &= \int_a^b f(r(t)) \cdot \left\| \frac{dr}{dt} \right\| dt \\
\iint_\Sigma f \cdot dR &= \iint_D f(R(u, v)) \cdot (T_u \times T_v)\, dA \quad \text{[Flux integral]} \\
\iint_\Sigma f dS &= \iint_D f(R(u, v)) \|T_u \times T_v\|\, dA
\end{aligned}$$

There is also the alternative evaluation of a surface integral of a vector field f given by $\iint_\Sigma f \cdot dR = \iint_\Sigma (f \cdot n) dS$ for a surface with outward normal n.