Lecture 25 - Monday June 1st

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Key words: Simple closed curve, Green's theorem, Divergence, Divergence theorem, Incompressible fluid, Outward normal

25.1 Green's Theorem.

Green's Theorem.

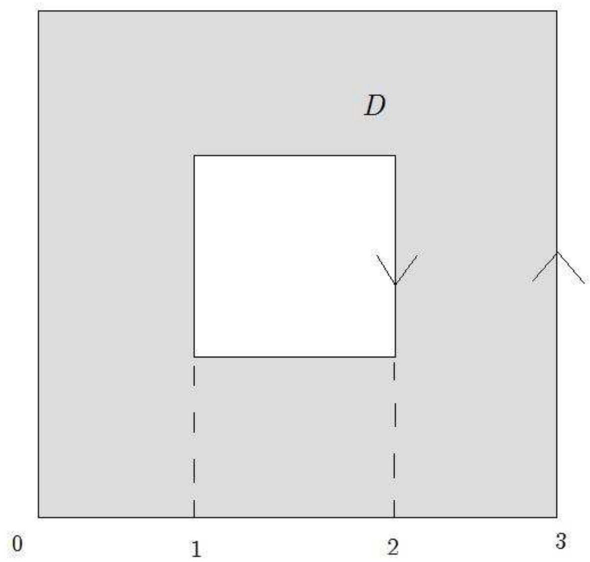
> Let γ be a simple closed curve in the plane and let D be the region it encloses. Suppose γ is oriented counterclockwise, and let $g, h : D \to \mathbb{R}$ are $C^1(D)$ functions. Then
>
> $$\int_\gamma g dx + h dy = \iint_D (h_x - g_y) dA$$

Green's Theorem holds for more general regions than those bounded by closed simple curves. As long as we can cut up the given region into regions bounded by closed simple curves in such a way that all the integrals of pieces of curves inside the region cancel out, Green's Theorem applies. For a general region D whose boundary consists of disjoint oriented closed simple curves $\gamma_1, \gamma_2, \dots, \gamma_n$, Green's Theorem applies provided that for each curve γ_i, the outward normal to γ_i is outside of the region D – in other words, walking around γ_i in the given orientation, the region D must be on the left as we walk around γ. This is why we can apply Green's Theorem with the inner curve in the above picture is oriented clockwise. If say $\gamma_1, \gamma_2, \dots, \gamma_m$ are oriented so that D is to the left of them, and $\gamma_{m+1}, \gamma_{m+2}, \dots, \gamma_n$ are oriented so that D is to the right of them as we walk around the curves, then by Green's Theorem the general formula is

$$\int_\gamma f \cdot dr = \sum_{i=1}^{m} \iint_{D_i} (h_x - g_y) dA - \sum_{j=m+1}^{n} \iint_{D_j} (h_x - g_y) dA$$

where D_i is the region enclosed by γ_i. So we "subtract" the integrals over regions where the boundary curve has the "wrong" orientation.

Example 1. Find $\int_\gamma e^{y^2} dx + \sin(x^2) dy$ where γ is the boundary of $D = [0,3] \times [0,3] \backslash [1,2] \times [1,2]$ and the inner part of γ is oriented clockwise and the outer part of γ is oriented counterclockwise.



Region D and boundary curve γ

Solution. Clearly γ is not a simple curve: it consists of the boundaries of two squares, one inside the other. Nevertheless Green's Theorem holds because D is always to the left in the given orientation of γ so we have a single formula:

$$\int_\gamma g dx + h dy = \iint_D (g_x - f_y) dA.$$

Now $g_y = 2ye^{y^2}$ and $h_x = 2x\cos(x^2)$ so the integral is

$$\iint_D 2xe^{x^2} dxdy - \iint_D 2y\cos(y^2) dydx.$$

The best way to compute this is by substracting the integral over the inner square $[1,2] \times [1,2]$ from the integral over the whole square $[0,3] \times [0,3]$: so the first integral is

$$\iint_D 2xe^{x^2} dxdy = \int_0^3 \int_0^3 2xe^{x^2} dxdy - \int_1^2 \int_1^2 2xe^{x^2} dxdy = 3e^9 - e^4 + e - 3$$

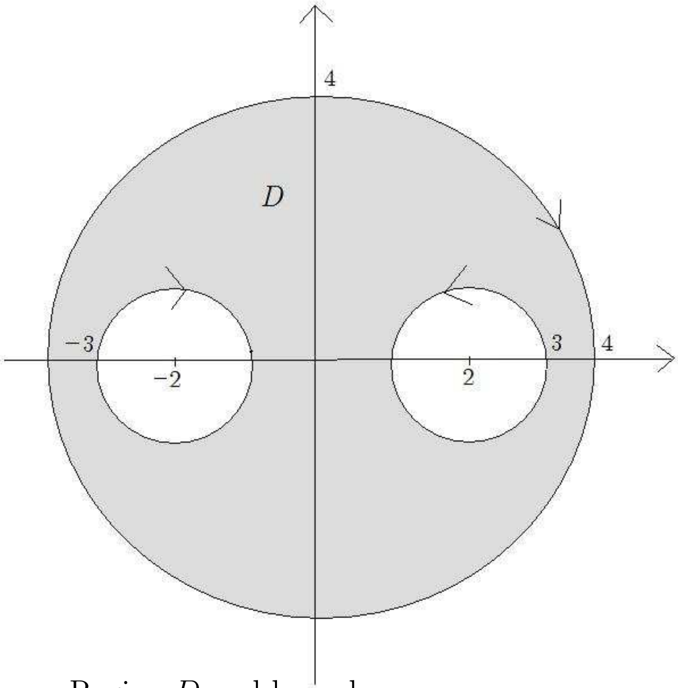
and similarly

$$\iint_D 2y\cos(y^2) dydx = \int_0^3 \int_0^3 2y\cos(y^2) dydx - \int_1^2 \int_1^2 2y\cos(y^2) dydx = 3\sin 9 - \sin 4 + \sin 1.$$

So we get

$$\int_\gamma g dx + h dy = 3e^9 - e^4 + e - 3 - 3\sin 9 + \sin 4 - \sin 1.$$

Example 2. Find $\int_\gamma f \cdot dr$ if $f(x, y) = (2xy, x^2 + y)$ and γ is the curve shown below.



Region D and boundary curve γ

Solution. The picture illustrates γ as the circle of radius four together with the two smaller circles of radius one centered at $(-2, 0)$ and $(0, 2)$. We cannot directly apply Green's Theorem since it is not true that the shaded region is always to the left as we walk along γ. Using the general formula, if D_1 is the region inside the large circle, D_2 is the region inside the left circle, and D_3 is the region inside the right circle, we get

$$\int_\gamma f \cdot dr = -\iint_{D_1} (1 - 2x) dA + \iint_{D_2} (1 - 2x) dA - \iint_{D_3} (1 - 2x) dA.$$

We leave the computation of the double integrals as an exercise – the answer is actually zero. We could have concluded this much earlier, since $\nabla \times f = 0$ and so f is a conservative vector field.

25.2 Divergence

First we define the notion of divergence of a vector field.

Definition.

The divergence of a vector field f, denoted $\text{div}(f)$ or $\nabla \cdot f$, is the scalar field

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}.$$

If the vector field f represents the velocity vector field of a fluid in $\mathbb{R}^3$, then the divergence of f is a measure of the rate of expansion per unit of volume of the fluid. If the divergence is positive, then the fluid is expanding, otherwise it is contracting. We imagine an open ball U containing a point (x, y, z) in the vector field with radius δ. All points in U are in motion according to the vector field, and after a short time t the points in U give a new region $U(t)$. We consider the rate of change of the volume as $\delta \to 0$ and the amount of time elapsed tends to zero – the resulting limit

$$\lim_{\delta, t \to 0} \frac{U'(t)}{U(0)} = \text{div} f(x, y, z)$$

is the divergence at (x, y, z). For example, the divergence of $f(x, y, z) = (x, y, z)$ is positive as we would expect (everything moving away from the origin) whereas $f(x, y) = (-y, z, x)$ has zero divergence and $f(x, y) = (-x, -y, -z)$ has negative divergence (everything moving into the origin). A fluid whose divergence is zero everywhere is sometimes called incompressible.

Example 1. Let f be a $C^2(\mathbb{R}^3)$ vector field. Then

$$\text{divcurl} f = \nabla \cdot (\nabla \times f) = 0.$$

Solution. This is a direct computation using the fact that since f is $C^2(\mathbb{R}^3)$, we have all components of f are $C^2(\mathbb{R}^3)$, and therefore all mixed partial derivatives of components of f are equal.

25.3 Divergence Theorem

The divergence theorem states that the density of matter in a bounded region of space can only change if there is flow of matter through the boundary of the region. It is also known as Gauss' Theorem.

Divergence Theorem in $\mathbb{R}^3$.

Let W be a simple region in $\mathbb{R}^3$ whose boundary is an oriented closed surface Σ with outward orientation. Let f be a $C^1(W)$ vector field. Then

$$\iiint \text{div} f dV = \iint_{\Sigma} f \cdot dR.$$

Recall that the integral on the right can also be written as

$$\iint_\Sigma f \cdot n dS$$

where n denotes the outward unit normal vector to Σ. A two-dimensional version of the divergence theorem, which follows from Green's Theorem, can be stated as follows. In this case a double integral is reduced to a line integral.

Divergence Theorem in $\mathbb{R}^2$.

Let γ be a simple closed curve in the plane with counterclockwise orientation and let D be the region it encloses. Let n be the outward unit normal to γ and let $f = (g, h)$ be a $C^1(D)$ vector field. Then

$$\iint_D \mathrm{div} f dA = \int_\gamma f \cdot dr.$$

Once again, we can replace the line integral with

$$\int_\gamma f \cdot n ds$$

where n is the outward unit normal to γ. We now check why this theorem is true via Green's Theorem. If $r(t) = (x(t), y(t))$ is an orientation preserving parametrization of γ, where $a \leq t \leq b$, then $r'(t) = (x'(t), y'(t))$ is tangent to γ at each point and therefore $(-y'(t), x'(t))$ is an outward normal to γ at every point on γ. So an outward unit normal is given by

$$n = \Big(\frac{-y'(t)}{\sqrt{x'(t)^2 + y'(t)^2}}, \frac{x'(t)}{\sqrt{x'(t)^2 + y'(t)^2}}\Big).$$

We now use this to work out the line integral of the scalar field $f \cdot n$ – which we said above is the same as the line integral of the vector field f – and apply Green's Theorem: if $f = (g, h)$, then

$$\int_\gamma f \cdot n ds = \int_a^b [g(r(t))y'(t) - h(r(t))x'(t)]dt = \int_\gamma [g dy - h dx].$$

By Green's Theorem,

$$\int_\gamma [g dy - h dx] = \iint_D (g_x + h_y) dA = \iint_D \mathrm{div} f dA$$

and this proves the divergence theorem. We now give some examples of applying the divergence theorem.

Example 1. If Σ is a closed surface and f is a $C^2(\Sigma)$ vector field, then

$$\iint_\Sigma (\nabla \times f) \cdot dR = \iiint_W \text{div}(\nabla \times f) dV$$

by the divergence theorem. Since $f \in C^2(\Sigma)$, we saw in the last section that $\text{div}(\nabla \times f) = 0$ and so the surface integral of the curl of a vector field over a closed surface is zero.

Example 2. If Σ is a closed surface containing a region W with outward orientation, and $f : \mathbb{R}^3 \to \mathbb{R}^3$ is tangent to Σ, then

$$\iiint_W \text{div} f dV = 0$$

by the divergence theorem. To see why this is true, note that

$$\iiint \text{div} f dV = \iint_\Sigma f \cdot dR = \iint_D (f \cdot n) dS$$

and since $f \cdot n = 0$ at every point of Σ, the integral is zero. This makes physical sense if f is the velocity field of a fluid – if f is tangent to Σ then no fluid is entering or leaving Σ, and therefore the total divergence for the region enclosed by Σ is zero.

Example 3. Evaluate

$$\iint_\Sigma f \cdot dR$$

where Σ is the surface of the unit hemisphere capped below by the disc $x^2 + y^2 \leq 1$, and $f(x, y, z) = (xz, x + y + z, \frac{1}{x} + \frac{1}{y} - z)$.

Solution. Clearly $\text{div} f = yz$ and therefore, by the divergence theorem

$$\iint_\Sigma f \cdot dR = \iiint_W z dV$$

where W is the interior of Σ. Translating to spherical co-ordinates we get

$$\int_0^1 \int_0^{\pi/2} \int_0^{2\pi} \rho^3 \sin\phi \cos\phi d\theta d\phi d\rho = \frac{\pi}{4}.$$

Example 4. If Σ is a closed surface with outward orientation in $\mathbb{R}^3$, and γ is a simple closed curve with counterclockwise orientation in the plane, then the volume of the region W enclosed by Σ and the area of the region D enclosed by γ are given respectively by

$$\frac{1}{2} \int_\gamma f \cdot nds$$
$$\frac{1}{3} \iint_\Sigma g \cdot ndS$$

where f is the vector field (x, y) in $\mathbb{R}^2$ and g is the vector field (x, y, z) in $\mathbb{R}^3$. To see why these formulae hold, we use the divergence theorem: for example, for the second integral, since g is $C^1(\mathbb{R}^3)$, we get

$$\iint_{\Sigma} g \cdot ndS = \iiint_W \text{div} g dV = 3 \iiint_W 1 dV = 3\text{volume}(W)$$

since $\text{div} g = 3$. We establish the first formula for the area of D similarly using the divergence theorem in $\mathbb{R}^2$.