Lecture 26 - Wednesday June 3rd

jacques@ucsd.edu

KEY WORDS: CIRCULATION, VORTICITY VECTOR, STOKES' THEOREM

26.1 Stokes' Theorem

There is a second interpretation of the curl of a vector field when the vector field represents the velocity field of a fluid. For such a vector field f, the curl $\nabla \times f$ at each point is exactly twice the angular velocity vector of a solid body which approximates the motion of the fluid near that point. If the vector field is conservative – so that $\text{curl} f = 0$ – then the vector field is also called irrotational, which means that a small solid body floating in the fluid tends only to exhibit lateral motion and no angular motion.

More generally, $(\nabla \times f) \cdot n$ denotes the tendency of the fluid to rotate around a normal axis n: precisely it is the circulation of the fluid per unit area at a given point on a surface perpendicular to the axis n. The curl is often called the vorticity vector. If γ is an oriented curve in a fluid with velocity field f, then the line integral

$$\int_\gamma f \cdot dr$$

is referred to as the circulation of f around γ. It is negative if the fluid tends to flow in the opposite direction to the curve γ, and zero if the fluid tends to flow perpendicular to γ, and positive if the fluid tends to flow with γ. Stokes' Theorem relates the circulation of a fluid around a curve γ forming the boundary of a surface with the tendency to rotate around normal axes to the surface:

Stokes' Theorem.

Let Σ be an oriented surface with boundary γ and let f be $C^1(\Sigma)$. Then

$$\iint_\Sigma (\nabla \times f) \cdot dR = \int_\gamma f \cdot dr.$$

Recall that

$$\iint_\Sigma (\nabla \times f) \cdot dR = \iint_\Sigma (\nabla \times f) \cdot n dS$$

and so the integral on the left can be considered as the total circulation of the fluid around normal axes to the surface Σ. Stokes' Theorem has the following consequence: if γ is a simple closed curve and f is a vector field, then the flux of the curl of f through any oriented surface Σ with boundary γ inheriting the orientation from Σ is the same regardless of what the surface Σ is. In particular, any vector field which is a curl of some other vector field has the same flux through all surfaces Σ with boundary γ.

Example 1. Let Σ denote the unit hemisphere with outward orientation and boundary γ given by $x^2 + y^2 = 1$. Suppose fluid flows across the hemisphere according to a velocity field $f(x, y, z) = (x, y, xyz)$. Determine

$$\iint_\Sigma (\nabla \times f) \cdot dR.$$

Solution. We use Stokes' Theorem

$$\iint_\Sigma (\nabla \times f) \cdot dR = \int_\gamma f \cdot dr.$$

Now we can parametrize γ as $r(\theta) = (\cos\theta, \sin\theta)$ for $0 \leq t < 2\pi$ and on this curve the vector field f is $g(\theta) = (\cos\theta, \sin\theta)$. Therefore

$$\iint_\Sigma (\nabla \times f) \cdot dR = \int_0^{2\pi} 0 d\theta = 0.$$

We expected the integral to be zero since f is perpendicular to γ at every point, and so the total circulation around γ – which is the line integral – must be zero.

Example 2. Let Σ denote the portion of the surface $z = x^2 + y^2$ below the plane $z = 2$ with downward orientation and let $f(x, y, z) = (3y, -x(z + 1), -yz^2)$. Evaluate

$$\iint_\Sigma (\nabla \times f) \cdot dR.$$

Solution. Since Σ has downward orientation, the boundary of Σ is the curve $x^2 + y^2 = 2$ with counterclockwise orientation. By Stokes' Theorem,

$$\iint_\Sigma (\nabla \times f) \cdot dR = \int_\gamma f \cdot dr.$$

Now we can parametrize γ by $r(\theta) = (\sqrt{2}\cos\theta, \sqrt{2}\sin\theta, 2)$ for $0 \leq \theta \leq 2\pi$ in which case $f = (3\sqrt{2}\sin\theta, -3\sqrt{2}\cos\theta, -4\sqrt{2}\sin\theta)$. Then since $r'(\theta) = (-\sqrt{2}\sin\theta, \sqrt{2}\cos\theta, 0)$, we get

$$\int_\gamma f \cdot dr = \int_0^{2\pi} (3\sqrt{2}\sin\theta, -3\sqrt{2}\cos\theta, -4\sqrt{2}\sin\theta) \cdot (-\sqrt{2}\sin\theta, \sqrt{2}\cos\theta, 0) d\theta = -12\pi.$$

We could also have computed the surface integral directly. First note that

$$\nabla \times f = (x - z^2, 0, -4 - z).$$

Taking the parametrization $R(u, v) = (u, v, u^2 + v^2)$ where $(u, v) \in D$ and $D = \{(u, v) : u^2 + v^2 \leq 2\}$, we get $T_u = (1, 0, 2u)$ and $T_v = (0, 1, 2v)$ so $T_u \times T_v = (-2u, -2v, 1)$. Furthermore $\nabla \times f = (u - (u^2 + v^2)^2, 0, -4 - u^2 - v^2)$. Therefore

$$\iint_\Sigma (\nabla \times f) \cdot dR = \iint_D \Big(-2u^2 + 2u(u^2 + v^2)^2 - 4 - u^2 - v^2\Big) dudv.$$

Changing to polar co-ordinates, we get

$$\int_0^{2\pi} \int_0^{\sqrt{2}} \left(-2r^2 \cos^2\theta + 2r^5 \cos\theta - 4 - r^2\right) r dr d\theta = -12\pi.$$

Example 3. Find the flux of a constant vector field (a, b, c) through the surface consisting of the co-ordinate planes and the plane $x+y+z=1$, with outward orientation, and where $x \geq 0, y \geq 0$ and $z \geq 0$.

Solution. We notice $f = \nabla \times g$ where $g = (bz, cx, ay)$. Therefore, by Stokes' Theorem, the flux is

$$\text{flux} = \iint_\Sigma (\nabla \times g) \cdot dR = \int_\gamma g \cdot dr$$

where γ is oriented counterclockwise and is the boundary of the triangle consisting of the lines $y = 0$, $y = 1 - x$ and $x = 0$. Let these curves be γ_1, γ_2 and γ_3 respectively. Parametrize these curves as follows:

$$\begin{aligned} \gamma_1 &: r(x) = (x, 0, 0) \text{ for } 0 \leq x \leq 1 \\ \gamma_2 &: r(x) = (1 - x, x, 0) \text{ for } 0 \leq x \leq 1 \\ \gamma_3 &: r(x) = (0, 1 - y, 0) \text{ for } 0 \leq y \leq 1 \end{aligned}$$

Then

$$\begin{aligned} \int_{\gamma_1} g \cdot dr &= \int_0^1 (0, cx, 0) \cdot (x, 0, 0) dx = 0 \\ \int_{\gamma_2} g \cdot dr &= \int_0^1 (0, cx, a(1-x)) \cdot (1 - x, x, 0) dx = \int_0^1 cx^2 dx = \frac{1}{3}c \\ \int_{\gamma_3} g \cdot dr &= \int_0^1 (0, 0, ay) \cdot (0, 1 - y, 0) dy = 0. \end{aligned}$$

Therefore the flux is $\frac{1}{3}c$.