Lecture 3 : Friday April 3rd

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Key concepts : Derivative, Differentiability
Know how to find derivative from definition and using usual methods

3.1 Further examples of non-existent limits

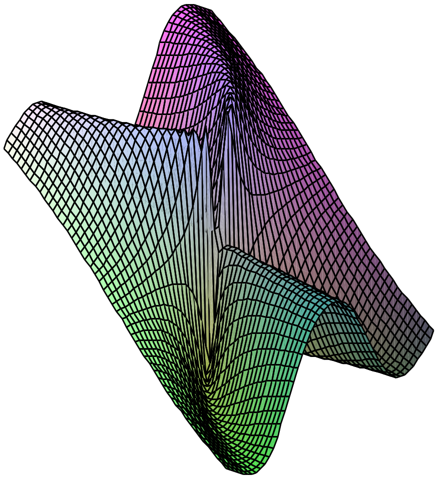
Example 1. Prove that the limit as $(x, y) \to 0$ of $f(x, y) = (x + y + x^2)/(x - y)$ does not exist.

Solution. Well let's put $y = mx$ to get $f(x, mx) = (1 + m)/(1 - m) + x/(1 - m)$ for $m \neq 1$. As $x \to 0$, this is $(1+m)/(1-m)$ which depends on m. For example if $m = 0$ we get 1 whereas if $m = -1$ we get 0. Therefore the original limit does not exist.

The next example is important because it shows that trying lines $y = mx$ to show that a limit as $(x, y) \to (0, 0)$ does not exist might fail. The point is, even if all limits $\lim_{x\to 0} f(x, mx)$ are equal (regardless of the value of m) this does not mean $\lim_{(x,y)\to(0,0)} f(x, y)$ exists. Instead, we try a different curve through $(0, 0)$, in the next example, the curve we try is $y = m\sqrt{x}$ for different values of m.

Example 2. Prove that the limit of $xy^2/(x^2 + y^4)$ as $(x, y) \to 0$ does not exist.

Solution. If we put $y = mx$ we get $mx^3/(x^2 + m^4x^4) = mx/(1 + m^4x^2)$. Now this limit as $x \to 0$ exists, so we can't conclude that the original limit does not exist. Therefore we select a different path, say $y = m\sqrt{x}$: in this case we get the function $m^2x^2/(x^2+m^4x^2) = m^2/(1 + m^4)$. If $m = 1$ we get $1/5$ and if $m = 0$ we get 0, so the original limit does not exist. A picture of $f(x, y) = xy^2/(x^2 + y^4)$ is shown below.



$$f(x, y) = \frac{xy^2}{x^2 + y^4}$$

3.2 Derivatives

Recall that for one-variable functions $f : \mathbb{R} \to \mathbb{R}$, the definition of the first derivative $\frac{df}{dx}(a)$ or $f'(a)$ is explicitly in terms of a limit:

$$\frac{df}{dx}(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.$$

In practice, to find $f'(a)$ we work out a general functional formula for $f'(x)$ and then substitute $x = a$. For example, if $f(x) = e^x$ then we know $f'(x) = e^x$ and so $f'(0) = 1$. Geometrically this means that the tangent line to $f(x)$ has slope $f'(0) = 1$ at $x = 0$. We shall generalize this to functions of several variables.

For functions $f : \mathbb{R}^n \to \mathbb{R}$, we write $f(x)$ instead of $(x_1, x_2, \ldots, x_n)$. There are now n co-ordinates with respect to which a derivative can be found, and we would wish that this represents the slope of the surface described by $f(x)$ in the direction of that co-ordinate.

Definition of partial derivative.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be defined on an open set $U \subset \mathbb{R}^n$ and let $a \in U$. Then the partial derivative of f with respect to x_j at the point a, if the limit below exists, is defined by

$$\frac{\partial f}{\partial x_j}(a) = \lim_{h\to 0} \frac{f(a_1, a_2, \ldots, a_{j-1}, a_j + h, a_{j+1}, \ldots, a_n) - f(a)}{h}.$$

Sometimes the derivative $\partial f/\partial x$ is denoted f_x and $\partial f/\partial x_j$ is denoted f_j.

3.3 Examples

Example 1. Let $f(x, y) = \sqrt{|xy|}$. Then

$$\frac{\partial f}{\partial x} = \frac{1}{2}\sqrt{\frac{|y|}{|x|}} \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{1}{2}\sqrt{\frac{|x|}{|y|}}$$

provided that $x \neq 0$ for the first derivative and $y \neq 0$ for the second derivative. So far we don't need to use the definition of derivatives to compute derivatives of f. But suppose we want $\frac{\partial f}{\partial x}(0, 0)$. We have to use the definition:

$$\frac{\partial f}{\partial x}(0, 0) = \lim_{h\to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0.$$

Similarly this holds for the derivative with respect to y. Therefore both partial derivatives are zero at $(0, 0)$, despite the fact that we cannot substitute $x = 0$ or $y = 0$ into the derivatives $\partial f/\partial x$ and $\partial f/\partial y$ respectively.

Example 2. Let $f(x,y) = e^{xy}$. Then $f_x = ye^{xy}$ and $f_y = xe^{xy}$. In both cases we can substitute $(x,y) = (0,0)$ to get that $f_x(0,0) = 0 = f_y(0,0)$.

The general rule for partial derivatives $\partial f/\partial x_j$ is to treat all variables apart from x_j as a constant and then use the usual rules of differentiation to find $\partial f/\partial x_j$. Let's do one final example.

Example 3. Let $f(x,y) = xy + x/y$. Then $f_x = y + 1/y$ and $f_y = x - x/y^2$. In this case $f_x(0,0)$ and $f_y(0,0)$ are not defined, so to see if the derivatives exist we use the definition:

$$f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h}.$$

This means the limit does not exist since $f(0,0)$ is not even defined. So f has no derivatives at $(0,0)$.

3.4 Differentiability

A function $f : \mathbb{R} \to \mathbb{R}$ is differentiable at a if and only if $f'(a)$ exists. For functions $f : \mathbb{R}^n \to \mathbb{R}$, differentiability at a point $a \in \mathbb{R}^n$ is not defined quite so simply, since it is not enough that all the partial derivatives f_j exist at a for all the usual rules of differentiation to hold.

Definition of differentiability.

> Let $f : \mathbb{R}^n \to \mathbb{R}$ be defined on an open set $U \subset \mathbb{R}^n$ and let $a \in U$. Then f is differentiable at a if all partial derivatives $f_j(a)$ exist and
>
> $$\lim_{h \to 0} \frac{f(x) - f(a) - \sum_{j=1}^{n} f_j(a)(x_j - a_j)}{d(a,x)} \to 0.$$

The notation $\sum_{j=1}^{n} f_j(a)(x_j - a_j)$ means add up $f_j(a)(x_j - a_j)$ for each value of j from one to n. For example, if $n = 2$ we get

$$\sum_{j=1}^{2} f_j(a)(x_j - a_j) = f_1(a)(x_1 - a_1) + f_2(a)(x_2 - a_2).$$

The important fact about the definition of differentiability is that it is not sufficient that all the derivatives $f_j(a)$ exist. Later we will see that this ensures that derivatives enjoy all of the usual properties that we would want them to have, such as the chain rule and product rule and so on. This will also be important for geometric reasons, in terms of defining tangents to surfaces in higher dimensions.