Lecture 5 : Wednesday April 8th

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Key concepts : Vector valued function, gradient, differentiability, properties, chain rule

5.1 Derivatives of vector-valued functions

We have not until now discussed derivatives of functions $f : \mathbb{R}^n \to \mathbb{R}^m$. In order to state the rules, we need some notation. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ assigns to each vector $x = (x_1, x_2, \ldots, x_n)$ a new vector $(f_1(x), f_2(x), \ldots, f_m(x))$. The derivative of f if it exists can be defined in terms of limits. However, for our purposes, we only need to know that

$$f_j(x) = (f_{1j}(x), f_{2j}(x), \ldots, f_{mj}(x)).$$

Remember $f_j(x)$ means the derivative of f with respect to x_j. So $f_{1j}(x)$ is the derivative of $f_1(x)$ with respect to x_j. Thus the derivative is found by taking derivatives of each of the components of $f(x)$.

Example. Let $f(x, y) = (x + y, x/y, x - y)$. This is a function $f : \mathbb{R}^2 \to \mathbb{R}^3$. Then

$$f_1(x, y) = (1, 1/y, 1) \quad \text{and } f_2(x, y) = (1, -x/y^2, -1).$$

The gradient of a function $f : \mathbb{R}^n \to \mathbb{R}^m$ where $f = (f_1, f_2, \ldots, f_m)$ is defined by the matrix ∇f whose ijth entry is f_{ij}. This is an $m \times n$ matrix. We write $\nabla f(a)$ to denote the matrix ∇f each of whose entries is evaluated at the point a.

Example. If $f(x, y) = (x + y, x/y, x - y)$, then ∇f is a 3×2 matrix and

$$\nabla f = \begin{pmatrix} 1 & 1 \\ 1/y & -x/y^2 \\ 1 & -1 \end{pmatrix}.$$

Furthermore,

$$\nabla f(0, 1) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & -1 \end{pmatrix}.$$

Example. If $f(x, y, z) = (x + y, x + z, y + z)$, then ∇f is a 3×3 matrix.

$$\nabla f = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

In the case $m = 1$, ∇f is just the usual gradient of the function $f(x_1, x_2, \ldots, x_n)$: it is the vector of partial derivatives $\partial f/\partial x_i$. We say that : $\mathbb{R}^n \to \mathbb{R}^m$ is differentiable at a if $\nabla f(a)$ exists and

$$\lim_{x \to a} \frac{d(f(x) - f(a), \nabla f(a) \cdot (x - a))}{d(x, a)} = 0.$$

Here we are writing $f(x) - f(a)$ as a column vector - namely an $m \times 1$ vector, and $\nabla f(a) \cdot (x - a)$ is the dot product of the $m \times n$ matrix $\nabla f(a)$ with the $n \times 1$ column vector $(x - a)$.

Example. Let $f(x, y) = (|xy|^{1/2}, y)$. To check whether this is differentiable at $(0, 0)$, we know already that

$$\nabla f(0, 0) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

In fact we already check that the partial derivatives of $|xy|^{1/2}$ at $(0, 0)$ are all zero. Now to check whether f is differentiable, we see

$$\nabla f(0, 0) \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ y \end{pmatrix}.$$

Also

$$f(x, y) - f(0, 0) = \begin{pmatrix} |xy|^{1/2} \\ y \end{pmatrix}.$$

The distance between $f(x, y) - f(0, 0)$ and $\nabla f(0, 0) \cdot (x, y)$ is exactly $|xy|^{1/2}$. So finally we just check

$$\lim_{(x,y) \to (0,0)} \frac{|xy|^{1/2}}{\sqrt{x^2 + y^2}} = 0.$$

But if we put $y = mx$ we get

$$\lim_{(x,y) \to (0,0)} \frac{m^{1/2}|x|}{(1 + m)^{1/2}|x|} = \frac{m^{1/2}}{(1 + m)^{1/2}}$$

and this is not zero. Therefore f is not differentiable.

Example. Let $f(x, y, z) = (xyz, x + y + z)$. Then

$$\nabla f = \begin{pmatrix} yz & xz & xy \\ 1 & 1 & 1 \end{pmatrix}.$$

It is clear that f is differentiable at $(0, 0, 0)$, but let's check it anyway.

$$\nabla f(0, 0, 0) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

Now

$$\nabla f(0,0,0) \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ x+y+z \end{pmatrix}.$$

The distance between $f(x,y,z) - f(0,0,0)$ as a column vector and the vector we just found is $|xyz|$. We now need

$$\lim_{(x,y,z)\to(0,0,0)} \frac{|xyz|}{\sqrt{x^2+y^2+z^2}} = 0.$$

We can prove this using the ϵ-δ definition of limits. Alternatively, we could say that since $x^2+y^2+z^2 \geq x^2$,

$$0 \leq \frac{|xyz|}{\sqrt{x^2+y^2+z^2}} \leq \frac{|xyz|}{|x|} \leq |yz|$$

and since $\lim |yz| = 0$ as $(x,y,z) \to (0,0,0)$, so we must also have that the limit above is zero. We are really using the squeeze theorem here. So our conclusion is that f is differentiable at $(0,0,0)$.