Lecture 6 : Friday April 10th

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Key concepts : Vector valued function, gradient, differentiability, properties, chain rule

6.1 Basic Properties of derivatives.

For functions of one variable $f : \mathbb{R} \to \mathbb{R}$, the basic properties of derivatives stem from the basic properties of limits, namely the sum, product and quotient rules, as well as the chain rule. The difference for functions of more than one variable is that for these properties to hold, the function is required to be differentiable. If the function is not differentiable then the rules can fail. Here are the basic properties of derivatives. For all of the following rules, $f, g : \mathbb{R}^n \to \mathbb{R}^m$ are defined on an open set U and differentiable functions at a point $a \in U$. For the quotient rule, g is non-zero on U.

- Constant multiple rule: $cf(x)$ is differentiable at a and

$$\nabla(cf)(a) = c\nabla f(a).$$

- Sum rule: $f(x) + g(x)$ is differentiable at a and

$$\nabla(f + g)(a) = \nabla f(a) + \nabla g(a).$$

- Product rule: for $m = 1$, $f(x) \cdot g(x)$ is differentiable at a and

$$\nabla(f \cdot g)(a) = \nabla f(a) \cdot g(a) + f(a) \cdot \nabla g(a).$$

- Quotient rule: for $m = 1$, $f(x)/g(x)$ is differentiable at a and

$$\nabla \left(\frac{f}{g}\right)(a) = (\nabla f(a) \cdot g(a) - f(a) \cdot \nabla g(a))/g(a)^2.$$

It is worthwhile thinking about the product rule for functions $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^n \cdot \mathbb{R}^m$. If we define the product $f \cdot g$ as $(f_1 g_1, f_2 g_2, \ldots, f_m g_m)$ when $f = (f_1, f_2, \ldots, f_m)$ and $g = (g_1, g_2, \ldots, g_m)$, can we write $\nabla(f \cdot g)$ in terms of ∇f and ∇g?

6.3 The Chain Rule

The chain rule is slightly more complicated for functions of more than one variable:

The chain rule.
Let $f : \mathbb{R}^m \to \mathbb{R}^p$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ be functions where the domain of g is an open set U and the domain of f is the range V of g. Suppose that g is differentiable at $a \in U$ and f is differentiable at $g(a) \in V$. Then $f \circ g$ is differentiable at a and

$$\nabla(f \circ g)(a) = \nabla f(g(a)) \cdot \nabla g(a).$$

It is important to notice here that $\nabla f(g(a)) \cdot \nabla g(a)$ denotes the multiplication of the matrices $\nabla f(g(a))$ and $\nabla g(a)$. This makes sense since ∇f is a $p \times m$ matrix and ∇g is an $m \times n$ matrix. In this case $\nabla(f \circ g)$ is a $p \times n$ matrix.