Lecture 8

Taylor's Theorem

Functions $f:\mathbb{R}\to\mathbb{R}$

Single Variable Taylor's Theorem

Let $f:\mathbb{R}\to\mathbb{R}$ be a function which has n continuous derivatives on the closed interval $[a,x]$ and suppose the $(n+1)^{\text{th}}$ derivative of f exists on the open interval (a,x). Then there exists a real number $\xi\in(a,x)$ such that

$$f(x) = \sum_{k=0}^{n} f^{(k)}(a)\frac{(x-a)^k}{k!} + f^{(n+1)}(\xi)\frac{(x-a)^{n+1}}{(n+1)!}$$

- The last term is denoted by $R_n(x,a)$ and called the Lagrange form of the remainder.
- The sum is denoted $T_n(x,a)$ and is called the Taylor series of order n for f at a.

Functions $f : \mathbb{R} \to \mathbb{R}$

☐ Example 1

If $f(x) = \log(1+x)$ then we compute

$$T_n(x,0) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{n-1} x^n}{n}$$

$$R_n(x,0) = \frac{(-1)^n x^{n+1}}{(n+1)(1+\xi)^{n+1}}$$

where $0 < \xi < x$. Therefore we get a good approximation for $\log(1+x)$ if $0 \le x \le 1$ for

$$|R_n(x,0)| \le \frac{|x|^{n+1}}{(n+1)} \le \frac{1}{n+1}$$

and this tends to zero.

Functions $f:\mathbb{R}^n \to \mathbb{R}$

- To state Taylor's Theorem for several variables properly, we need some notation.

- If $f:\mathbb{R}^n \to \mathbb{R}$ is a function of variables $(x_1, x_2, \ldots, x_n)$ and z is any unordered list of x_is, then $f_z(x)$ means the derivative of f with respect to all the entries in z.

- We write $|z|$ for the number of elements of z.

Example 1

Consider polynomial $f(x_1, x_2, x_3) = x_1 + x_1 x_2 x_3^2$ and let $z = x_1 x_3 x_3 x_2$. Then

$$f_z = f_{x_1 x_3 x_3 x_2} = 2$$

The order of derivatives does not matter since (by the preceding lecture) $f \in C^\infty(\mathbb{R}^3)$.

Furthermore $|z| = 4$ in this example.

Functions $f : \mathbb{R}^n \to \mathbb{R}$

Example 2

If $f : \mathbb{R} \to \mathbb{R}$ then $f_z(x)$ is just the $|z|^{\text{th}}$ derivative of f with respect to x.

- Suppose that x_i appears d_i times in z. Then

$$z! = d_1!\, d_2! \cdots d_n!$$

using the convention that $0! = 1$.

- It can be seen that $z!$ is precisely $|z|!$ divided by the number of ways to re-order the entries of z.

- Suppose that x_i appears d_i times in z and let $y \in \mathbb{R}^n$. Then we write y^z to denote

$$y_1^{d_1} y_2^{d_2} \cdots y_n^{d_n}$$

Functions $f:\mathbb{R}^n \to \mathbb{R}$

☐ Example 3

If $z = xxx...xx$ then $z! = |z|!$.

☐ Example 4

In the example with $z = x_1x_3x_3x_2$ we have $z! = 1!2!1! = 2$. Alternatively, z can be re-ordered in the 12 ways below, so we get $z! = |z|!/12 = 2$:

(x_1,x_3,x_3,x_2), (x_2,x_3,x_3,x_1), (x_1,x_2,x_3,x_3),
(x_2,x_1,x_3,x_3), (x_3,x_3,x_2,x_1), (x_3,x_3,x_1,x_2),
(x_3,x_1,x_3,x_2), (x_3,x_2,x_3,x_1), (x_3,x_1,x_2,x_3),
(x_3,x_2,x_1,x_3), (x_1,x_3,x_2,x_3), (x_2,x_3,x_1,x_3).

Furthermore if $y = (x_1 - a_1, x_2 - a_2, x_3 - a_3)$ then

$$y^z = (x_1 - a_1)(x_2 - a_2)(x_3 - a_3)^2.$$

Functions $f:\mathbb{R}^n \to \mathbb{R}$

Multivariable Taylor's Theorem

Let $f:\mathbb{R}^n \to \mathbb{R}$ be a function which is N times continuously differentiable on the closed ball $\mathbb{B}[a,x]$ and suppose the $(N+1)^{\text{th}}$ derivative of f exists on the open ball $\mathbb{B}(a,x)$. Then there exists a vector $\xi \in \mathbb{B}(a,x)$ such that

$$f(x) = \sum_{|z|=0}^{N} f_z(a)\frac{(x-a)^z}{z!} + \sum_{|z|=N+1} f_z(\xi)\frac{(x-a)^z}{z!}$$

$$= T_N(x,a) + R_N(x,a)$$

☐ Example 5

We find the third order Taylor formula for $f(x,y) = e^{xy}$ at the origin. It is given by

$$T_3(x,0) = \sum_{|z|=0}^{3} f_z(0)\frac{x^z}{z!}.$$

Functions $f:\mathbb{R}^n \to \mathbb{R}$

- First note that $f_z(0) = f(0) = 1$ if $|z| = 0$. This indicates taking no derivatives of f.

- For $|z| = 1$ the contribution to $T_3(x,y)$ is zero.

- For $|z| = 2$ we have

$$\begin{aligned}
&\text{for } z = xx : f_z(0) = f_{xx}(0) = 0,\ z! = 2 \\
&\text{for } z = xy : f_z(0) = f_{xy}(0) = 1,\ z! = 1 \\
&\text{for } z = yy : f_z(0) = f_{yy}(0) = 0,\ z! = 2
\end{aligned}$$

  So the contribution to $T_3(x,y)$ here is xy. Since z is an unordered list, note that $z = xy$ is the same as $z = yx$.

- For $|z| = 3$ the contribution is zero.

- So finally we get $T_3(x,y) = 1 + xy$.

Actually if we recall that the Taylor series for e^t is

$$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots$$

then it makes sense that

$$e^{xy} = 1 + xy + \frac{x^2y^2}{2!} + \frac{x^3y^3}{3!} + \cdots$$

and this agrees with the Taylor series of order three we found before.

☐ Example 6

Find the Taylor series for $f(x, y) = \log(1 + x + y)$ of order two at the origin.

▷ First note that $f_z(0) = f(0) = 0$ if $|z| = 0$. This indicates taking no derivatives of f.

▷ For $|z| = 1$ we have $z = x$ and $z = y$.
for $z = x$: $f_z(0) = f_x(0) = 1,\ z! = 1$
for $z = y$: $f_z(0) = f_y(0) = 1,\ z! = 1$
So the contribution to $T_3(x, y)$ here is $x + y$.

▷ For $|z| = 2$ we have

$$\begin{aligned} &\text{for } z = xx \; : \; f_z(0) = f_{xx}(0) = -1, \; z! = 2 \\ &\text{for } z = xy \; : \; f_z(0) = f_{xy}(0) = -1, \; z! = 1 \\ &\text{for } z = yy \; : \; f_z(0) = f_{yy}(0) = -1, \; z! = 2 \end{aligned}$$

So the contribution to $T_2(x, y)$ here is

$$(-1) \cdot \frac{x^2}{2} + (-1) \cdot xy + (-1) \frac{y^2}{2} = -\tfrac{1}{2}(x+y)^2.$$

▷ So $T_2(x, y) = x + y - \frac{1}{2}(x+y)^2$.

☐ Example 7

We find the third term of the Taylor series for $f(x, y) = 1/(1 + x^2 y)$ at $(x, y) = 0$. The possible values of z are, as can be checked by a computation,

$z = xxx$	$z = yyy$	$z = xyy$	$z = xxy$
$z! = 6$	$z! = 6$	$z! = 2$	$z! = 2$
$f_z(0) = 0$	$f_z(0) = 0$	$f_z(0) = 0$	$f_z(0) = -2$.

Therefore the third term is $\dfrac{1}{2}(-2)x^2 y = -x^2 y$.