Practice Midterm Examination

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Time: 40 minutes

No notes allowed

All questions carry equal weight

Question 1.

(a) Show that $xy \leq \frac{1}{2}(x^2 + y^2)$.

(b) Use part (a) and the ϵ-δ definition of limits to show

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2 + y^2}} = 0.$$

a) Since $(x-y)^2 \geq 0$

$$x^2 - 2xy + y^2 \geq 0$$

$$\frac{1}{2}(x^2+y^2) \geq xy$$

as required.

b) $$\left|\frac{xy}{\sqrt{x^2+y^2}}\right| < \epsilon$$

$$\Leftarrow \left|\frac{\frac{1}{2}(x^2+y^2)}{\sqrt{x^2+y^2}}\right| < \epsilon \quad \text{by (a)}$$

$$\Leftarrow \sqrt{x^2+y^2} < 2\epsilon \quad \text{Put } \delta = 2\epsilon$$

Question 2.

Define what it means for a function $f(x) : \mathbb{R}^n \to \mathbb{R}$ to be differentiable at a point a. Then prove that the function $f(x,y) = |xy|^{1/2}$ is not differentiable at $(x,y) = (0,0)$.

See class notes – we did this

Question 3.

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(u, v) = uv$ and let $g : \mathbb{R}^2 \to \mathbb{R}^2$ be define by $g(x, y) = (y, x)$. If $\phi(x, y) = f(g(x, y))$, use the chain rule to find $\frac{\partial \phi}{\partial x}$.

Chain rule

$$\nabla(f \circ g)(x,y)$$

$$= \nabla f(g(x,y)) \circ \nabla g(x,y)$$

$$\nabla f = \left(\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}\right)$$

$$= (v, u)$$

$$\nabla f(g(x,y)) = \nabla f(y,x)$$

$$= (x, y)$$

$$\nabla g(x,y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

So by the chain rule

$$\nabla(f \circ g)(x,y) = \nabla \phi(x,y)$$

$$= (x,y) \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = (y,x)$$

Therefore $\frac{\partial \phi}{\partial x} = y$.

Question 4.

Compute the second order Taylor formula for $f(x, y) = \log(1 + x + y)$ about the point $(x, y) = (0, 0)$.

Taylor's Theorem Not Examinable