Practice Midterm Examination

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Time: 40 minutes

No notes allowed

All questions carry equal weight

Question 1.

State precisely the ϵ-δ definition of $\lim_{x \to a} f(x) = L$ for a function $f : \mathbb{R}^n \to \mathbb{R}$. Then prove using the ϵ-δ definition of limits that

$$\lim_{(x,y)\to(0,0)} \sin(x^2 + y^2) = 0.$$

Definition of limit: BOOKWORK

(Be Precise!!)

$|\sin(x^2+y^2)| < \epsilon$

$\Leftarrow |x^2+y^2| < \epsilon$ since $|\sin z| \le |z|$ always

$\Leftarrow \sqrt{x^2+y^2} < \sqrt{\epsilon}$

so take $\delta = \sqrt{\epsilon}$ in definition.

Question 2.

Find the direction of steepest increase of the function $f(x,y) = (x+y)e^{xy}$ from the origin. What is the equation of the tangent hyperplane to the surface $z = f(x,y)$ at the origin?

The direction of steepest descent is

$$\nabla f(0,0) = \left(y(x+y)e^{xy} + e^{xy},\ x(x+y)e^{xy} + e^{xy} \right)_{\substack{x=0 \\ y=0}}$$

$$= (1,1).$$

The tangent hyperplane is

$$z = f(0,0) + x\frac{\partial f}{\partial x}(0,0) + y\frac{\partial f}{\partial y}(0,0)$$

$$= f(0,0) + \underbrace{\nabla f(0,0)}_{= (1,1).} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= x + y$$

Question 3.

State precisely the chain rule for determining the gradient of $\nabla(f \circ g)(a)$ where $a \in \mathbb{R}^m$ and $f : \mathbb{R}^n \to \mathbb{R}^p$ and $g : \mathbb{R}^m \to \mathbb{R}^n$ are functions. Then determine $\nabla(f \circ g)(1)$ when $f : \mathbb{R}^2 \to \mathbb{R}^3$ is defined by $f(x, y) = (x, y, xy)$ and when $g : \mathbb{R} \to \mathbb{R}^2$ is defined by $g(z) = (z, 1/z)$.

Chain rule: Bookwork

$$\nabla f = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ y & x \end{pmatrix}$$

$$\nabla f(g(1)) = \nabla f(1,1)$$

since $g(1) = (1,1)$

$$\text{so } \nabla f(g(1)) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\nabla g = \begin{pmatrix} 1 \\ -1/z^2 \end{pmatrix}$$

$$\nabla g(1) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\nabla(f \circ g)(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

Question 4.

Find all second order partial derivatives for the function $f : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$f(x, y, z) = (1 + x)(1 + y)(1 + z).$$

$f_x = (1+y)(1+z)$

$f_y = (1+x)(1+z)$

$f_z = (1+x)(1+y)$

$f_{xx} = f_{yy} = f_{zz} = 0$

f is $C^\infty(\mathbb{R}^3)$ so only need f_{xy}, f_{xz}, f_{yz}

$f_{xy} = (1+z)$ $\quad f_{yz} = (1+x)$

$f_{xz} = (1+y)$