

Practice Midterm Examination

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Time: 40 minutes

No notes allowed

All questions carry equal weight

Question 1. TOUGH

Define the distance $d(x, y)$ for $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ and define the open ball $B(a, r)$ of radius r centered at point $a \in \mathbb{R}^n$. Define what it means for a set $U \subseteq \mathbb{R}^n$ to be open. Then use the definition to show the following:

- (a) Show that the set of real numbers which have a 1 in their decimal representation is an open set
- (b) Show that the set of real numbers which do not have a 1 in their decimal representation is not an open set.

(a) Let D be the set and let $x \in D$.

Then $x = x_n 10^n + x_{n-1} 10^{n-1} + \dots$

and for some i , $x_i = 1$. Take a ball of radius $\frac{1}{2}(10^{n-i})$ around x — all points $y \in \mathbb{R}$ with $|y - x| < \frac{1}{2} 10^{n-i}$.

We claim $y \in D$. Well note $y_i = x_i = 1$ otherwise $|y - x| \geq 10^{n-i}$. So $y \in D$.

(b) Let D^c be this set. If $x \in D^c$, put $y_i = x_i$ for $i \leq N$ and $y_{N+1} = 1$.

Then $y \notin D^c$ but y is as close to x as we want.

BOOK
WORK

Question 2.

Prove that the following limit does not exist

$$\lim_{(x,y,z) \rightarrow (0,0,0)} \frac{xyz}{x + y^2 + z^3}$$

Let $z = -x^{1/3}$ to get

$$\lim_{(x,y) \rightarrow (0,0)} \frac{-x^{4/3}y}{y^2} = \lim_{(x,y) \rightarrow (0,0)} \frac{-x^{4/3}}{y}$$

Now let $y = mx^{4/3}$ $m \neq 0$
to get

$$\lim_{x \rightarrow 0} \frac{-1}{m} = \frac{-1}{m}$$

Depends on m so limit fails.

Question 3.

Find the directional derivative of $f(x, y, z) = xyz$ at $(1, 1, 1)$ in the direction of the unit vector $(1/\sqrt{2}, 1/\sqrt{2}, 0)$. Then find the direction in which $f(x, y, z)$ is steepest and find the directional derivative in that direction.

Find $\nabla f(1, 1, 1) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) = \sqrt{2}$

Steepest direction is

$$\nabla f(1, 1, 1) = (1, 1, 1)$$

Directional derivative

$$\nabla f(1, 1, 1) \cdot \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \sqrt{3}$$

↓ unit vector
in direction
of $(1, 1, 1)$

Question 4.

Let $f(x, y, z) = xyz$. Determine $\nabla f(x, y, z)$, and then determine $\nabla(\nabla f(x, y, z))$.

$$\nabla f(x, y, z) = (yz, xz, xy)$$

$$\nabla(\nabla f(x, y, z))$$

$$= \begin{pmatrix} \nabla yz \\ \nabla xz \\ \nabla xy \end{pmatrix}$$

$$= \begin{pmatrix} 0 & z & y \\ z & 0 & x \\ y & x & 0 \end{pmatrix}.$$