

# Final Examination

Math 20E – Vector Calculus

Instructor – J. Verstraete

Allotted time – 3 hours

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Answers are to be written clearly and legibly  
Calculators are allowed  
State clearly any theorems used without proof  
Total 50 points

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### Question 1.

(a) Define  $\lim_{x \rightarrow a} f(x) = L$  when  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ .

(b) Prove that the following limit does not exist:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2 + y^2}.$$

## Question 2.

Find the *minimum value* of the function  $f(x, y, z) = x^2 + y^2 + z^2$  subject to the constraint  $2x + y + z = 1$ .

### Question 3.

State *Fubini's Theorem* for a continuous function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  on a rectangle  $[a, b] \times [c, d]$ . Then evaluate the double integral

$$\int_0^{2\pi} \int_0^{2\pi} \sin(x + y^2) dy dx.$$

### Question 4.

Consider the transformation from Cartesian coordinates  $(X, Y)$  to coordinates  $(u, v)$  given by the following equations:

$$X = u^2 \cos^4 v \quad \text{and} \quad Y = u^2 \sin^4 v$$

valid for  $u > 0$  and  $0 < v < \pi/2$ . Show that the *Jacobian determinant* for this transformation is  $(u \sin 2v)^3$ .

### Question 5.

(a) Use the transformation in the last question to show that

$$\int_0^{\infty} \int_0^{\infty} e^{-\sqrt{X} - \sqrt{Y}} dY dX = 4.$$

(b) Then find

$$\int_0^{\infty} e^{-\sqrt{t}} dt.$$

### Question 6.

- (a) Prove that the vector field  $f(x, y) = (ye^{xy}, xe^{xy})$  is *conservative*.
- (b) Evaluate the line integral  $\int_{\gamma} f \cdot dr$  when  $\gamma$  is a curve joining  $(0, 0)$  to  $(1, 1)$ .

### Question 7.

(a) State the *divergence theorem*.

(b) Let  $\Sigma$  denote the surface of the box  $[1, 2] \times [1, 2] \times [1, 2]$  with outward orientation. Determine

$$\iint_{\Sigma} f \cdot dR$$

where

$$f(x, y, z) = (x^2 + y^y, y^2 + z^z, z^2 + x^x).$$

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