Practice Final Examination #2

Math 20E – Vector Calculus

Instructor – J. Verstraete

Allotted time – 3 hours

Answers are to be written clearly and legibly
Calculators are allowed
State clearly any theorems used without proof
Total 50 points

Question 1.

(a) State the definition of $\lim_{x \to a} f(x) = L$ where $f : \mathbb{R}^n \to \mathbb{R}$.

(b) Then show that $\lim_{(x,y) \to (0,0)} \frac{e^{xy}-1}{x^2+y^2}$ does <u>not</u> exist.

Question 2.

Find the minimum value of the function $f(x, y) = \frac{1}{x^2} + \frac{1}{y^2}$ subject to the constraint $xy = 1$.

Question 3.

(a) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a function whose second order partial derivatives exist on $\mathbb{R}^3$. Prove that

$$\nabla \cdot \nabla f = f_{xx} + f_{yy} + f_{zz}.$$

(b) If the second order partial derivatives of f are continuous on $\mathbb{R}^3$, show that

$$\nabla \times \nabla f = 0.$$

Question 4.

(a) State <u>Fubini's Theorem</u> for improper integrals.

(b) Let $D = [0, 2\pi] \times [0, 2\pi]$. Show that the integral

$$\iint_D \frac{\sin x}{y} dA$$

does not exist, and explain why this does not contradict Fubini's Theorem.

Question 5.

(a) Prove that the Jacobian determinant for a change of variable to spherical co-ordinates

$$x = \rho\cos\theta\sin\phi \quad y = \rho\sin\theta\sin\phi \quad z = \rho\cos\phi$$

where $\rho > 0$, $0 \leq \phi < \pi$ and $0 \leq \theta < 2\pi$ is $\rho^2\sin\phi$.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be a function and suppose $\int_0^1 f(t)dt = \alpha$. Show that

$$\iiint_D \frac{f(\sqrt{x^2+y^2+z^2})}{\sqrt{x^2+y^2+z^2}}dV = 4\pi\alpha$$

where D is the unit ball $\{(x, y, z) : x^2 + y^2 + z^2 \leq 1\}$.

Question 6.

Let γ denote any smooth curve joining $(0,0)$ to $(1,1)$, and let $f(x,y)$ be the vector field $(ye^{xy}+1, xe^{xy}+1)$. Determine

$$\int_\gamma f \cdot dr.$$

Question 7.

(a) State the divergence theorem.

(b) Then use Question 3 to determine the surface integral $\int\int_\Sigma \nabla f \cdot dR$ where Σ is the surface of the unit cube $[0,1] \times [0,1] \times [0,1]$ with outward orientation, and

$$f(x,y,z) = (e^{yz}, \tan(xz), z^2).$$

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