Math 20E Solutions to Assignment 1 jacques@ucsd.edu

This assignment carries a total of twenty points. The assignments contribute ten percent to your final grade.

Question 1. Show that $D = \{(x, y)|x \neq 0 \text{ and } y \neq 0\}$ is open.

Solution. For any point $(x, y) \in D$, the distance from (x, y) to the x-axis is $|x|$ and the distance to the y-axis is $|y|$. Now since D is the set of all points in $\mathbb{R}^2$ not on the x-axis and not on the y-axis, the open ball of radius $\min\{|x|, |y|\}$ centered at (x, y) is entirely contained in D. So every point in D is contained in an open ball which is contained in D, which shows D is open. **I will accept answers which assume** $D = \{(x, y)|(x, y) \neq (0, 0)\}$.

Question 2. Compute the following limit if it exists

$$\lim_{(x,y)\to(0,0)} \frac{e^{xy} - 1}{y}.$$

Solution 1. If $z = xy$, we have

$$\frac{e^{xy} - 1}{y} = x \cdot \frac{e^z - 1}{z}.$$

Now clearly. $\lim_{(x,y)\to 0} x = 0$. Furthermore, by the substitution rule,

$$\lim_{(x,y)\to(0,0)} \frac{e^{xy} - 1}{xy} = \lim_{z\to 0} \frac{e^z - 1}{z}.$$

By l'Hopital's Rule,

$$\lim_{z\to 0} \frac{e^z - 1}{z} = \lim_{z\to 0} e^z = 1.$$

By the product rule for limits,

$$\lim_{(x,y)\to(0,0)} \frac{e^{xy} - 1}{y} = \left(\lim_{(x,y)\to 0} x\right) \cdot \left(\lim_{z\to 0} \frac{e^z - 1}{z}\right) = 0 \cdot 1 = 0.$$

Solution 2. If z is sufficiently small, then $|e^z - 1| \leq 2|z|$ since the slope of $e^z - 1$ at $z = 0$ is 1. We use this and the ϵ-δ definition of limits to show that the given limit is zero. For any $\epsilon > 0$.

$$\left|\frac{e^{xy} - 1}{y}\right| \leq \left|\frac{2xy}{y}\right| \leq 2|x| \leq 2\sqrt{x^2 + y^2}.$$

Since $\sqrt{x^2 + y^2} = d((x, y), (0, 0))$, we get

$$\sqrt{x^2 + y^2} < \frac{1}{2}\epsilon \Rightarrow \left|\frac{e^{xy} - 1}{y}\right| < \epsilon.$$

So putting $\delta = \epsilon/2$ in the definition of the limit, we see that the limit is zero.

Solution 3. This is basically the same as solution 2: we use the squeeze theorem: if $|g(x, y)| \leq |f(x, y)| \leq |h(x, y)|$ and

$$\lim_{(x,y)\to(0,0)} g(x, y) = 0 = \lim_{(x,y)\to(0,0)} h(x, y)$$

then the limit of f as $(x, y) \to (0, 0)$ is also L. In the present case, we let

$$f(x, y) = \frac{e^{xy} - 1}{y} \quad g(x, y) = 0 \quad h(x, y) = 2x.$$

For then as we checked in solution 2, $|g(x, y)| \leq |f(x, y)| \leq |h(x, y)|$. Since g and h both have zero limits, so must f.

Question 3. Can $xy/(x^2 + y^2)$ be made continuous by suitably defining it at $(0, 0)$?

Solution. If we could make this into a continuous function f with $f(0, 0) = L$, then we would need

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2} = f(0, 0) = L.$$

The question is does such an L exist? If we let $y = mx$, then we get

$$\lim_{(x,y)\to(0,0)} \frac{mx^2}{x^2 + m^2x^2} = \frac{m}{1 + m^2}.$$

This means no value L exists, since for $m = 0$ we get zero and for $m = 1$ we get $\frac{1}{2}$. In other words,

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2} \quad \text{does not exist}$$

so we can't make it continuous at $(0, 0)$.

Question 4. Use the ϵ-δ definition of limits to prove $x^2 \to 4$ as $x \to 2$. Give another proof using the product rule.

Solution. We have to show that for all $\epsilon > 0$ there exists $\delta > 0$ such that if $|x-2| < \delta$ then $|x^2 - 4| < \epsilon$. If $\epsilon \geq 1$, then take $\delta = \frac{1}{5}$. This works because

$$|x-2| < \frac{1}{5} \to |x-2||x+2| < \frac{1}{5} \cdot \left(\frac{1}{5} + 4\right) < \epsilon.$$

If $\epsilon < 1$, we will show that $\delta = \frac{1}{5}\epsilon$ works. For if $|x-2| < \delta$, then definitely $|x+2| < \delta + 4 < 5$. Therefore

$$|x-2| < \delta \to |x-2||x+2| < 5\delta \leftrightarrow |x^2 - 4| < \epsilon.$$

This gives the limit, by definition. For the product rule, just notice

$$\lim_{x \to 2} x^2 = \left(\lim_{x \to 2} x\right) \cdot \left(\lim_{x \to 2} x\right) = 2 \cdot 2 = 4.$$