

This assignment carries a total of twenty points. The assignments contribute ten percent to your final grade.

---

**Question 1.** Compute the matrix of partial derivatives of  $f(x, y, z) = (x - y, y + z)$ .

**Solution.**

$$\nabla f(x, y, z) = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

**Question 2.** Consider the function

$$f(x, y) = \begin{cases} \frac{xy^2}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{Otherwise} \end{cases}$$

Show (a) that  $\partial f/\partial x$  and  $\partial f/\partial y$  exist at  $(0, 0)$  and (b) If  $g(t) = (at, bt)$  for constants  $a$  and  $b$ , then  $f \circ g$  is differentiable and  $(f \circ g)'(0) = ab^2/(a^2 + b^2)$ , but  $\nabla f(0, 0) \cdot g'(0) = 0$ .

**Solution 2.** For (a) we use the definition of partial derivatives:

$$\frac{\partial f}{\partial x}(0, 0) = \lim_{h \rightarrow 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{0 - 0}{h} = 0.$$

So  $f_x(0, 0) = 0$  and similarly  $f_y(0, 0) = 0$ . For (b), note that

$$f(g(t)) = \frac{ab^2t^3}{(a^2 + b^2)t^2} = \frac{ab^2t}{a^2 + b^2}$$

and so the derivative of  $f(g(t))$  with respect to  $t$  is  $ab^2/(a^2 + b^2)$ , as required. On the other hand,

$$\nabla f(0, 0) \cdot g'(0) = (0, 0) \cdot (0, 0) = 0.$$

**Question 3.** A function  $u = f(x, y)$  with continuous second partial derivatives satisfying Laplace's equation

$$u_{xx} + u_{yy} = 0$$

is called a harmonic function. Show that  $u = x^3 - 3xy^2$  is harmonic.

**Solution 3.** Since  $u_{xx} = 6x$  and  $u_{yy} = 6x$ ,  $u_{xx} - u_{yy} = 0$  and so  $u$  is harmonic.

**Question 4.** Determine the second-order Taylor formula for  $e^{x+y}$  at  $(0, 0)$ .

**Solution.** Recall

$$T_N(x, y) = \sum_{|z|=0}^N f_z(0) \frac{x^z}{z!}$$

is the  $N$ th order Taylor formula. For  $N = 2$  we get

$$T_2(x, y) = \sum_{|z|=0}^2 f_z(0) \frac{x^z}{z!}.$$

For  $|z| = 0$  we get  $z! = 1 = x^z$  and  $f_z(0) = f(0) = 1$ . For  $|z| = 1$  we have  $z = x$  or  $z = y$ . So we get  $z! = 1$  in each case and  $f_x(0) = 1$  and  $f_y(0) = 1$ . So far we have

$$T_1(x, y) = 1 + x + y.$$

Finally, for  $|z| = 2$  we get  $z! = 2! = 2$  when  $z = xx$ ,  $z! = 2! = 2$  when  $z = yy$ , and  $z! = 1!1! = 1$  when  $z = xy$ . Also  $x^z$  in each case is  $x^2$ ,  $y^2$  and  $xy$ . Also  $f_z(0) = 1$  in all cases, so

$$T_2(x, y) = 1 + x + y + \frac{1}{2}x^2 + \frac{1}{2}y^2 + xy.$$

**Question 5.** Find the critical points of  $f(x, y) = xy + 1/x + 1/y$  and then determine whether they are local maxima, local minima, or saddle points.

**Solution.** Clearly  $\nabla f(x, y) = (y - 1/x^2, x - 1/y^2)$  so the critical points are points  $(x, y)$  where  $y = 1/x^2$  and  $x = 1/y^2$ , or where  $\nabla f$  does not exist. Now  $\nabla f$  does not exist when  $x = 0$  or  $y = 0$ . But since  $f$  is not defined there, we don't have to worry about those points. Now suppose  $x \neq 0$  and  $y \neq 0$ . Then  $\nabla f = 0$  implies  $y = y^4$  and so  $y(1 - y^3) = 0$ . Therefore  $y = 1$  which gives  $x = 1$ , so  $(1, 1)$  is a critical point. The Hessian matrix at  $(x, y)$  is

$$H_f(x, y) = \begin{pmatrix} 2/x^3 & 1 \\ 1 & 2/y^3 \end{pmatrix}.$$

At  $(1, 1)$ , this is

$$H_f(1, 1) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Since  $\det(H_1) = 2$  and  $\det(H_2) = 3$ ,  $H_f(1, 1)$  is positive definite. Therefore  $f$  has a local minimum at  $(x, y) = (1, 1)$ .