Solutions to Assignment 3.

Question 1. Let $n > 2$ be an integer and set $f(x, y) = ax^n + by^n$ where $ac \neq 0$. Determine the nature of the critical points of f.

Solution. Since $\nabla f = (anx^{n-1}, bny^{n-1})$ and $n > 2$, this is zero if and only if $x = y = 0$. The Hessian matrix for f is

$$H_f(x, y) = \begin{pmatrix} an(n-1)x^{n-2} & 0 \\ 0 & bn(n-1)y^{n-2} \end{pmatrix}$$

which is the zero matrix at $(0, 0)$. So we cannot use the second derivative test to determine the nature of the critical points of f. Instead we just check values of $f(x, y)$ near $(0, 0)$. First consider the case that <u>n is even</u>. Then if a and c are positive, $(0, 0)$ represents a minimum for f because $f(x, y) > 0$ for all $(x, y) \neq (0, 0)$. If a and c are negative, then $(0, 0)$ represents a maximum for f because $f(x, y) < 0$ for all $(x, y) \neq (0, 0)$. If a and c are of different sign, then $(0, 0)$ represents neither a minimum nor a maximum because $f(x, 0)$ and $f(0, y)$ have different signs whereas $f(0, 0) = 0$. Next consider the case that <u>n is odd</u>. In this case, we can make ax^n as negative as we like and by^n as positive as we like around $(0, 0)$, so we can find points (x, y) as close as we like to $(0, 0)$ so that $f(x, y) < 0$ or $f(x, y) > 0$. Therefore if n is odd, $(0, 0)$ is neither a local minimum nor a local maximum.

Question 2. Find the absolute maximum and minimum of $f(x, y, z) = x + yz$ on $x^2 + y^2 + z^2 \leq 1$.

Solution. $\nabla f = (1, z, y)$ which is never zero, so f has no critical points in the interior of the given region. Now on the boundary $x^2 + y^2 + z^2 = 1$, we use Lagrange Multipliers: we set $\nabla f + \lambda \nabla g = 0$ where $g(x, y, z) = 1 - x^2 - y^2 - z^2$. So we get

$$\begin{aligned} 1 - 2\lambda x &= 0 \\ z - 2\lambda y &= 0 \\ y - 2\lambda z &= 0 \end{aligned}$$

The last two equations give $y(1 - 4\lambda^2) = 0 = z(1 - 4\lambda^2)$.

Case 1. If $\lambda = \pm\frac{1}{2}$ then we get $x = \pm 1$ from the first equation. Since $x^2 + y^2 + z^2 = 1$ this means $y = z = 0$. So $(1, 0, 0)$ and $(-1, 0, 0)$ are possible extreme points on the boundary of the region.

Case 2. If $\lambda \neq \pm\frac{1}{2}$, then $y = z = 0$ and $x^2 + y^2 + z^2 = 1$ means $x = \pm 1$. So again we get $(1, 0, 0)$ and $(-1, 0, 0)$ are possible extreme points.

Then $(1, 0, 0)$ and $(-1, 0, 0)$ represent a maximum and minimum for f on the given region, since they are the only critical points of f on the region and we have a theorem that says that f has an absolute minimum and maximum on a closed bounded set – here the closed set is $x^2 + y^2 + z^2 \leq 1$.

Question 3. Investigate whether or not the system

$$\begin{aligned} u &= x + xyz \\ v &= y + xy \\ w &= z + 2x + 3z^2 \end{aligned}$$

can be solved for x, y, z in terms of u, v, w near $(x, y, z) = (0, 0, 0)$.

Solution. The inverse function theorem says that we can solve the system near $(0, 0, 0)$ if the determinant of the matrix

$$\begin{pmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{pmatrix}$$

is non-zero. This matrix is

$$\begin{pmatrix} 1 + yz & xz & xy \\ y & 1 + x & 0 \\ 2 & 0 & 1 + 6z \end{pmatrix}$$

At $(0, 0, 0)$ this matrix is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}.$$

Expanding along the first row, the determinant equals 1. Therefore the system is solvable near $(0, 0, 0)$.

Question 4. Compute the volume of the solid bounded by $z = \sin y$, the planes $x = 1$, $x = 0$, $y = 0$ and $y = \pi/2$ and the xy plane.

Solution. The region of integration is $\{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq \pi/2\}$. So the volume is (order of integration does not matter, by Fubini's Theorem)

$$\int_0^1 \int_0^{\pi/2} \sin y dy dx = \int_0^1 dx = 1.$$

Question 5. Find

$$\int_0^4 \int_{y/2}^2 e^{x^2} dx dy.$$

Solution. The function e^{x^2} has no explicit antiderivative, so we'd like to change the order of integration. Since e^{x^2} is continuous, Fubini's Theorem says we can. Now the given region is $\{(x, y) : 0 \leq y \leq 4, y/2 \leq x \leq 2\}$. But we can rewrite it as

$$\{(x, y) : 0 \leq x \leq 2, 0 \leq y \leq 2x\}$$

so the integral is

$$\int_0^2 \int_0^{2x} e^{x^2} dy dx = \int_0^2 2xe^{x^2} dx = \int_0^4 e^w dw = e^4 - 1$$

where we made the substitution $w = x^2$.