Solutions to Assignment 4.

Question 1. Calculate $\displaystyle\iint_R (x+y)^2 e^{x-y} dxdy$ where R is the region bounded by $x+y=1$, $x+y=4$, $x-y=-1$ and $x-y=1$.

Solution. The change of variables $u = x+y$ and $v = x-y$ is suggested. Then R becomes the region $[1,4] \times [-1,1]$ in uv-coordinates. The Jacobian determinant for this transformation is

$$\frac{1}{\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}} = -\frac{1}{2}.$$

Therefore

$$\iint_R (x+y)^2 e^{x-y} dxdy = \frac{1}{2}\int_1^4 \int_{-1}^1 u^2 e^v dvdu = 5(e - 1/e).$$

Question 2. Discuss whether the integral $\iint_D \frac{x+y}{x^2+2xy+y^2} dxdy$ exists when $D = [0,1] \times [0,1]$. If it exists, compute its value.

Solution. Note that $x^2 + 2xy + y^2 = (x+y)$ so the integral is really

$$\iint_D \frac{1}{x+y} dxdy = \int_0^1 \int_0^1 \frac{1}{x+y} dxdy = \int_0^1 (\ln(1+y) - \ln y) dy.$$

Since $\displaystyle\int \ln y \, dy = y \ln y - y + c$, and $\lim_{y \to 0^+} y \ln y = 0$, we get that the integral is $2\ln 2$. By Fubini's Theorem for improper integrals, the integral is $2 \ln 2$ since the integrand is non-negative on D.

Question 3. The image of the path $t \mapsto (\cos^3 t, \sin^3 t)$, $0 \leq t \leq 2\pi$ in the plane is a hypocycloid (see the book). Evaluate the integral of $F(x,y) = xi + yj$ around this curve.

Solution. If $r(t) = (\cos^3 t, \sin^3 t)$ then $r'(t) = (-3\cos^2 t \sin t, 3 \sin^2 t \cos t)$. Also $F(r(t)) = (\cos^3 t, \sin^3 t)$ so

$$\begin{aligned} \int_\gamma F \cdot dr &= \int_0^{2\pi} (\cos^3 t, \sin^3 t) \cdot (-3\cos^2 t \sin t, 3\sin^2 t \cos t) \, dt \\ &= 3\int_0^{2\pi} (\cos^5 t)(-\sin t) dt + 3\int_0^{2\pi} (\sin^5 t)(\cos t) dt \\ &= 0. \end{aligned}$$

Note that since $F = \nabla G$ where $G = \frac{1}{2}x^2 i + \frac{1}{2}y^2 j$, F is conservative so we knew the line integral would be zero.

Question 4. Show that the surface $x = 1/\sqrt{y^2+z^2}$, $1 \leq x < \infty$, can be filled but not painted.

Solution. We might as well consider the function $z = 1/\sqrt{x^2+y^2}$ for $1 \leq z < \infty$. The volume of the region W under this surface and above the xy-plane is $\iiint_W 1 dV$. In cylindrical co-ordinates, W is described by $r \leq 1$ (since $x^2+y^2 \leq 1$ comes from $1 \leq z < \infty$) and $0 \leq \theta \leq 2\pi$ and $1 \leq z < \frac{1}{r}$. So the volume is

$$\int_0^{2\pi} \int_0^1 \int_1^{1/r} r dz dr d\theta = \int_0^{2\pi} \int_0^1 (1-r) dr d\theta = \pi.$$

So the region can be filled. To say that it cannot be painted means it has infinite surface area. The surface area is

$$\iint_D \sqrt{1+z_x^2+z_y^2} dA$$

where $D = \{(x,y) : x^2+y^2 \leq 1\}$. Now $z_x = x/(x^2+y^2)^{3/2}$ and $z_y = y/(x^2+y^2)^{3/2}$. Converting to polar co-ordinates, the integral is

$$\int_0^{2\pi} \int_0^1 \sqrt{1+\frac{1}{r^4}} r dr d\theta \geq \int_0^{2\pi} \int_0^1 \frac{1}{r} dr d\theta.$$

This integral diverges since the inner integral in r is infinite. Therefore the surface area is infinite.

Question 5. Show that if S is a surface described implicitly as $F(x,y,z) = 0$ for $(x,y) \in D$, then

$$\iint_S \left|\frac{\partial F}{\partial z}\right| dS = \iint_D \sqrt{F_x^2+F_y^2+F_z^2} dx dy.$$

Solution. Recall from implicit differentiation that

$$F_x + F_z z_x = 0 \quad \text{ and } F_y + F_z z_y = 0.$$

Now if we parametrize S by $(x, y, z(x,y))$, then

$$T_x = (1, 0, z_x) \quad \text{ and } T_y = (0, 1, z_y).$$

Therefore

$$\begin{aligned}
\iint_S \left|\frac{\partial F}{\partial z}\right| dS &= \iint_D \sqrt{z_x^2 + z_y^2 + 1}\,|F_z|\,dxdy \\
&= \iint_D \sqrt{F_z^2 z_x^2 + F_z^2 z_y^2 + F_z^2}\,dxdy \\
&= \iint_D \sqrt{F_x^2 + F_y^2 + F_z^2}
\end{aligned}$$

using the formulas of implicit differentiation.