Practice Final Examination #1

Math 20E – Vector Calculus

Instructor – J. Verstraete

Allotted time – 3 hours

Answers are to be written clearly and legibly
Calculators are allowed
State clearly any theorems used without proof
Total 50 points

Question 1.

(a) Define $\lim_{x\to a} f(x) = L$ when $f : \mathbb{R}^n \to \mathbb{R}$.

(b) Prove that the following limit does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}.$$

(a) BOOKWORK

(b) Let $y = mx$. Then

$$\frac{xy}{x^2+y^2} = \frac{mx^2}{x^2+m^2x^2} = \frac{m}{1+m^2}$$

so $$\lim_{(x,mx)\to(0,0)} = \lim_{x\to 0} \frac{m}{1+m^2} = \frac{m}{1+m^2}$$

which depends on m. So the limit does not exist

Question 2.

Find the *minimum value* of the function $f(x,y,z) = x^2 + y^2 + z^2$ subject to the constraint $2x + y + z = 1$.

Use Lagrange Multiplies:

$$\phi(x,y,z,\lambda) = x^2 + y^2 + z^2 + \lambda(1 - 2x - y - z)$$

$$\nabla \phi = 0 \Rightarrow$$

$$\frac{\partial \phi}{\partial x} = 0 \Rightarrow 2x - 2\lambda = 0$$

$$\frac{\partial \phi}{\partial y} = 0 \Rightarrow 2y - \lambda = 0$$

$$\frac{\partial \phi}{\partial z} = 0 \Rightarrow 2z - \lambda = 0$$

So $y = \frac{\lambda}{2}$, $z = \frac{\lambda}{2}$, $x = \lambda$

Since $2x + y + z = 1$

we get $2\lambda + \frac{\lambda}{2} + \frac{\lambda}{2} = 1$

so $\lambda = 1/3$ which means

$$x = 1/3, \quad z = 1/6, \quad y = 1/6.$$

Now $f(1/3, 1/6, 1/6) = \frac{1}{9} + \frac{1}{36} + \frac{1}{36}$

$= \frac{1}{6}$ is the minimum of f

Question 3.

State *Fubini's Theorem* for a continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ on a rectangle $[a, b] \times [c, d]$. Then evaluate the double integral

$$\int_0^{2\pi} \int_0^{2\pi} \sin(x + y^2) dy dx.$$

FUBINI'S THEOREM : BOOKWORK

Since $\sin(x+y^2)$ is continuous on $[0, 2\pi] \times [0, 2\pi]$, Fubini's Theorem applies: we can interchange the order of integration:

$$\int_0^{2\pi} \int_0^{2\pi} \sin(x+y^2)\, dy\, dx$$

$$= \int_0^{2\pi} \int_0^{2\pi} \sin(x+y^2)\, dx\, dy$$

$$= \int_0^{2\pi} \left. -\cos(x+y^2) \right]_0^{2\pi} dy$$

$$= \int_0^{2\pi} \left(\cos y^2 - \cos(2\pi + y^2)\right) dy$$

$$= \int_0^{2\pi} 0\, dy = 0.$$

Question 4.

Consider the transformation from Cartesian coordinates (X, Y) to coordinates (u, v) given by the following equations:

$$X = u^2 \cos^4 v \quad \text{and} \quad Y = u^2 \sin^4 v$$

valid for $u > 0$ and $0 < v < \pi/2$. Show that the *Jacobian determinant* for this transformation is $(u \sin 2v)^3$.

Jacobian Matrix

$$\begin{pmatrix} \frac{\partial X}{\partial u} & \frac{\partial X}{\partial v} \\ \frac{\partial Y}{\partial u} & \frac{\partial Y}{\partial v} \end{pmatrix}$$

$$= \begin{pmatrix} 2u \cos^4 v & -4u^2 \cos^3 v \sin v \\ 2u \sin^4 v & 4u^2 \sin^3 v \cos v \end{pmatrix}$$

Jacobian determinant.

$$= 8u^3 \cos^5 v \sin^3 v + 8u^3 \cos^3 v \sin^5 v$$

$$= 8u^3 \sin^3 v \cos^3 v \left(\cos^2 v + \sin^2 v\right)$$

$$= 8u^3 \sin^3 v \cos^3 v$$

$$= \underrightarrow{(u \sin 2v)^3}$$

Question 5.

(a) Use the transformation in the last question to show that

$$\int_0^\infty \int_0^\infty e^{-\sqrt{X}\ -\sqrt{Y}}\ dY dX = 4.$$

(b) Then find

$$\int_0^\infty e^{-\sqrt{t}} dt.$$

$$\sqrt{X} = u\cos^2 v \qquad \sqrt{Y} = u\sin^2 v$$

so $$-\sqrt{X} - \sqrt{Y} = -u(\cos^2 v + \sin^2 v)$$

$$= -u$$

Using Q4 we get

$$\int_0^{\frac{\pi}{2}}\int_0^\infty e^{-u} \cdot (u\sin 2v)^3 \, du \, dv$$

$$= \left(\int_0^{\frac{\pi}{2}} (\sin 2v)^3 dv\right) \cdot \left(\int_0^\infty \underset{\substack{\downarrow \\ \text{BY PARTS} \\ \downarrow}}{u^3 e^{-u} du}\right)$$

$$= \int_0^{\frac{\pi}{2}} (\sin 2v)(1-\cos^2 2v)\, dv \cdot 6$$

$$= -\tfrac{1}{2}\cos 2v\Big]_0^{\frac{\pi}{2}} + \tfrac{1}{2}\cdot\tfrac{1}{3}\cos^3 2v\Big]_0^{\frac{\pi}{2}} \cdot 6$$

$$= \left(\left(\tfrac{1}{2}+\tfrac{1}{2}\right) - \tfrac{1}{6} - \tfrac{1}{6}\right) \cdot 6$$

$$= \tfrac{2}{3}\cdot 6 = 4.$$

(b) $$\int_0^\infty e^{-\sqrt{t}} dt = \sqrt{\int_0^\infty\int_0^\infty e^{-\sqrt{X}-\sqrt{Y}} dY dX} = 2.$$

Question 6.

(a) Prove that the vector field $f(x, y) = (ye^{xy}, xe^{xy})$ is *conservative.*

(b) Evaluate the line integral $\int_\gamma f \cdot dr$ when γ is a curve joining $(0, 0)$ to $(1, 1)$.

(a) We can show $\operatorname{curl}(f) = \nabla \times f = 0$ or we can find a potential function F:

$$\frac{\partial F}{\partial x} = ye^{xy} \Rightarrow F(x,y) = y \cdot \frac{e^{xy}}{y} + \text{constant}$$

$$\frac{\partial F}{\partial y} = xe^{xy} \Rightarrow F(x,y) = x \cdot \frac{e^{xy}}{y} + \text{constant}$$

So we can take $F(x,y) = e^{xy}$.

<u>Or</u> $\operatorname{curl}(f) = \nabla \times f = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ ye^{xy} & xe^{xy} & 0 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix}$

$$= \hat{k}\left[\frac{\partial}{\partial y}(ye^{xy}) - \frac{\partial}{\partial x}(xe^{xy})\right]$$

$$= \hat{k}\left(e^{xy} + xye^{xy} - (e^{xy} + xye^{xy})\right)$$

$$= \hat{k} \cdot 0 = 0$$

Therefore f is conservative.

(b) Since f is conservative, if F is any potential function then (using $F = e^{xy}$)

$$\int_\gamma f \cdot dr = F(1,1) - F(0,0)$$

$$= e^{1 \cdot 1} - e^{0 \cdot 0}$$

$$= e - 1$$

Question 7.

(a) State the *divergence theorem.*

(b) Let Σ denote the surface of the box $[1,2]\times[1,2]\times[1,2]$ with outward orientation. Determine

$$\iint_\Sigma f\cdot dR$$

where

$$f(x,y,z)=\left(x^2+y^y,y^2+z^z,z^2+x^x\right).$$

(a) BOOKWORK

(b) The divergence theorem gives

$$\iint_\Sigma f\cdot dR = \iiint_V \operatorname{div}(f)$$

$$= \int_1^2\int_1^2\int_1^2 (f_x + g_y + h_z)\,dx\,dy\,dz$$

$$= \int_1^2\int_1^2\int_1^2 \frac{\partial}{\partial x}\left(x^2+y^y\right) + \frac{\partial}{\partial y}\left(y^2+z^z\right) + \frac{\partial}{\partial z}\left(z^2+x^x\right)\,dx\,d$$

$$= \int_1^2\int_1^2\int_1^2 (2x+2y+2z)\,dx\,dy\,dz$$

$$= \int_1^2\int_1^2 (3+2y+2z)\,dy\,dz$$

$$= \int_1^2 (6+2z)\,dz$$

$$= \underset{\longrightarrow}{9}\ .$$

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