Practice Final Examination #2

Math 20E – Vector Calculus

Instructor – J. Verstraete

Allotted time – 3 hours

Answers are to be written clearly and legibly
Calculators are allowed
State clearly any theorems used without proof
Total 50 points

Question 1.

(a) State the definition of $\lim_{x\to a} f(x) = L$ where $f : \mathbb{R}^n \to \mathbb{R}$.

(b) Then show that $\lim_{(x,y)\to(0,0)} \frac{e^{xy}-1}{x^2+y^2}$ does not exist.

(a) BOOKWORK

(b) Put $y = mx$. Then

$$\lim_{(x,mx)\to(0,0)} \frac{e^{xmx}-1}{x^2+m^2x^2}$$

$$= \lim_{x\to 0} \frac{e^{mx^2}-1}{x^2(1+m^2)}$$

Use l'Hôpital's Rule

$$= \lim_{x\to 0} \frac{2mx e^{mx^2}}{2x(1+m^2)}$$

$$= \lim_{x\to 0} \frac{m e^{mx^2}}{1+m^2}$$

$$= \frac{m \cdot e^0}{1+m^2} = \frac{m}{1+m^2}$$

This depends on m so limit does not exist.

Question 2.

Find the <u>minimum value</u> of the function $f(x,y) = \frac{1}{x^2} + \frac{1}{y^2}$ subject to the constraint $xy = 1$.

Lagrange Multipliers

$$\phi(x,y,\lambda) = \frac{1}{x^2} + \frac{1}{y^2} + \lambda(1 - xy)$$

$$\nabla\phi = 0$$

$$\Rightarrow \frac{\partial\phi}{\partial x} = 0 \Rightarrow -\frac{2}{x^3} - \lambda y = 0$$

$$\Rightarrow \frac{\partial\phi}{\partial y} = 0 \Rightarrow -\frac{2}{y^3} - \lambda x = 0$$

so $\lambda = \dfrac{-2}{x^3 y} = \dfrac{-2}{y^3 x}$ so

$$x^3 y = y^3 x$$

i.e. $xy(x^2 - y^2) = 0$

so $x = 0$ or $y = 0$ or $x^2 = y^2$.

Clearly $x=0$ or $y=0$ is impossible since $xy = 1$.

So $x^2 = y^2$. Since $xy = 1$, we have $x^2 = y^2 = 1$

and so the minimum is

$$f(\pm1, \pm1) = \frac{1}{(\pm1)^2} + \frac{1}{(\pm1)^2} = 2.$$

Question 3.

(a) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a function whose second order partial derivatives exist on $\mathbb{R}^3$. Prove that

$$\nabla \cdot \nabla f = f_{xx} + f_{yy} + f_{zz}.$$

(b) If the second order partial derivatives of f are continuous on $\mathbb{R}^3$, show that

$$\nabla \times \nabla f = 0.$$

(a) By definition

$$\nabla \cdot \nabla f = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

$$= \frac{\partial}{\partial x}\frac{\partial f}{\partial x} + \frac{\partial}{\partial y}\frac{\partial f}{\partial y} + \frac{\partial}{\partial z}\frac{\partial f}{\partial z}$$

$$= f_{xx} + f_{yy} + f_{zz}.$$

(b)

$$\nabla \times \nabla f = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix}$$

$$= \hat{i}\,(f_{zy} - f_{yz}) - \hat{j}\,(f_{zx} - f_{xz}) + \hat{k}\,(f_{yx} - f_{xy}).$$

Since second order partial derivatives are continuous,

$$f_{xy} = f_{yx}, \quad f_{yz} = f_{zy}, \quad f_{xz} = f_{zx}$$

so $$\nabla \times \nabla f = 0\hat{i} - 0\hat{j} + 0\hat{k} = 0.$$

Question 4.

(a) State Fubini's Theorem for improper integrals.

(b) Let $D = [0, 2\pi] \times [0, 2\pi]$. Show that the integral

$$\iint_D \frac{\sin x}{y} dA$$

does not exist, and explain why this does not contradict Fubini's Theorem.

(a) BOOKWORK

(b)

$$\int_0^{2\pi} \int_0^{2\pi} \frac{\sin x}{y} \, dx \, dy$$

$$= \int_0^{2\pi} \frac{1}{y} \left(-\cos x\right]_0^{2\pi} dy$$

$$= \int_0^{2\pi} \frac{1}{y} \cdot 0 \, dy = 0 .$$

$$\int_0^{2\pi} \int_0^{2\pi} \frac{\sin x}{y} \, dy \, dx$$

$$= \int_0^{2\pi} \ln|y| \Big]_0^{2\pi} \sin x \, dx$$

$$= \int_0^{2\pi} \left(\ln 2\pi - \lim_{y \to 0^+} \ln|y|\right) \sin x \, dx$$

and $\lim_{y \to 0^+} \ln|y| = -\infty$ so the limit does not exist and neither does this integral.

Does not contradict Fubini since $\sin x$ can be negative on $[0, 2\pi]$.

Question 5.

(a) Prove that the Jacobian determinant for a change of variable to spherical co-ordinates

$$x = \rho\cos\theta\sin\phi \quad y = \rho\sin\theta\sin\phi \quad z = \rho\cos\phi$$

where $\rho > 0$, $0 \leq \phi < \pi$ and $0 \leq \theta < 2\pi$ is $\rho^2\sin\phi$.

(b) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function and suppose $\int_0^1 f(t)dt = \alpha$. Show that

$$\iiint_D \frac{f(\sqrt{x^2+y^2+z^2})}{\sqrt{x^2+y^2+z^2}}dV = 4\pi\alpha$$

where D is the unit ball $\{(x, y, z) : x^2 + y^2 + z^2 \leq 1\}$.

(a) BOOKWORK.

(b)

Question 6.

Let γ denote any smooth curve joining $(0,0)$ to $(1,1)$, and let $f(x,y)$ be the vector field $(ye^{xy}+1, xe^{xy}+1)$. Determine

$$\int_\gamma f \cdot dr.$$

The given vector field is <u>conservative</u> since

$$\nabla \times f = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ ye^{xy}+1 & xe^{xy}+1 & 0 \end{vmatrix}$$

$$= \hat{i}\cdot 0 - \hat{j}\, 0 + \hat{k}\left(\frac{\partial}{\partial x}(xe^{xy}+1) - \frac{\partial}{\partial y}(ye^{xy}+1)\right)$$

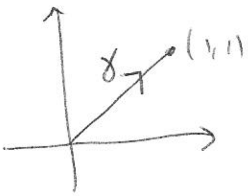
$$= \hat{k}\left(\left(e^{xy} + xye^{xy}\right) - \left(e^{xy} + xye^{xy}\right)\right)$$

$$= \hat{k}\, 0$$

$$= (0,0,0).$$

That means we can replace γ with an easier path from $(0,0)$ to $(1,1)$ and still get the same answer.

Take γ to be the straight line $x=t, y=t; \; 0 \le t \le 1$



Parametrization $r(t) = (t,t)$ so

$$\frac{dr}{dt} = (1,1) \text{ and}$$

$$\int_\gamma f \cdot dr = \int_0^1 \left(te^{t^2}+1, te^{t^2}+1\right)\cdot(1,1)\, dt$$

$$= \int_0^1 \left(2te^{t^2}+2\right) dt$$

$$= (e-1) + 2 = e+1$$

Question 7.

(a) State the divergence theorem.

(b) Then use Question 3 to determine the surface integral $\iint_\Sigma \nabla f \cdot dR$ where Σ is the surface of the unit cube $[0,1] \times [0,1] \times [0,1]$ with outward orientation, and

$$f(x,y,z) = (e^{yz}, \tan(xz), z^2).$$

$$f_1 \quad f_2 \quad f_3$$

(a) BOOKWORK

(b) Divergence theorem

$$\iint_\Sigma \nabla f \cdot dR = \iiint_W \nabla \cdot \nabla f \, dV$$

$$= \iiint_W (f_{1xx} + f_{2yy} + f_{3zz}) \, dV \text{ by Q3.}$$

Now $(f_1)_{xx} = (f_2)_{yy} = 0 \quad (f_3)_{zz} = 2$

So

$$= \int_0^1 \int_0^1 \int_0^1 2 \, dz \, dy \, dx$$

$$= 2.$$

$\longrightarrow$

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