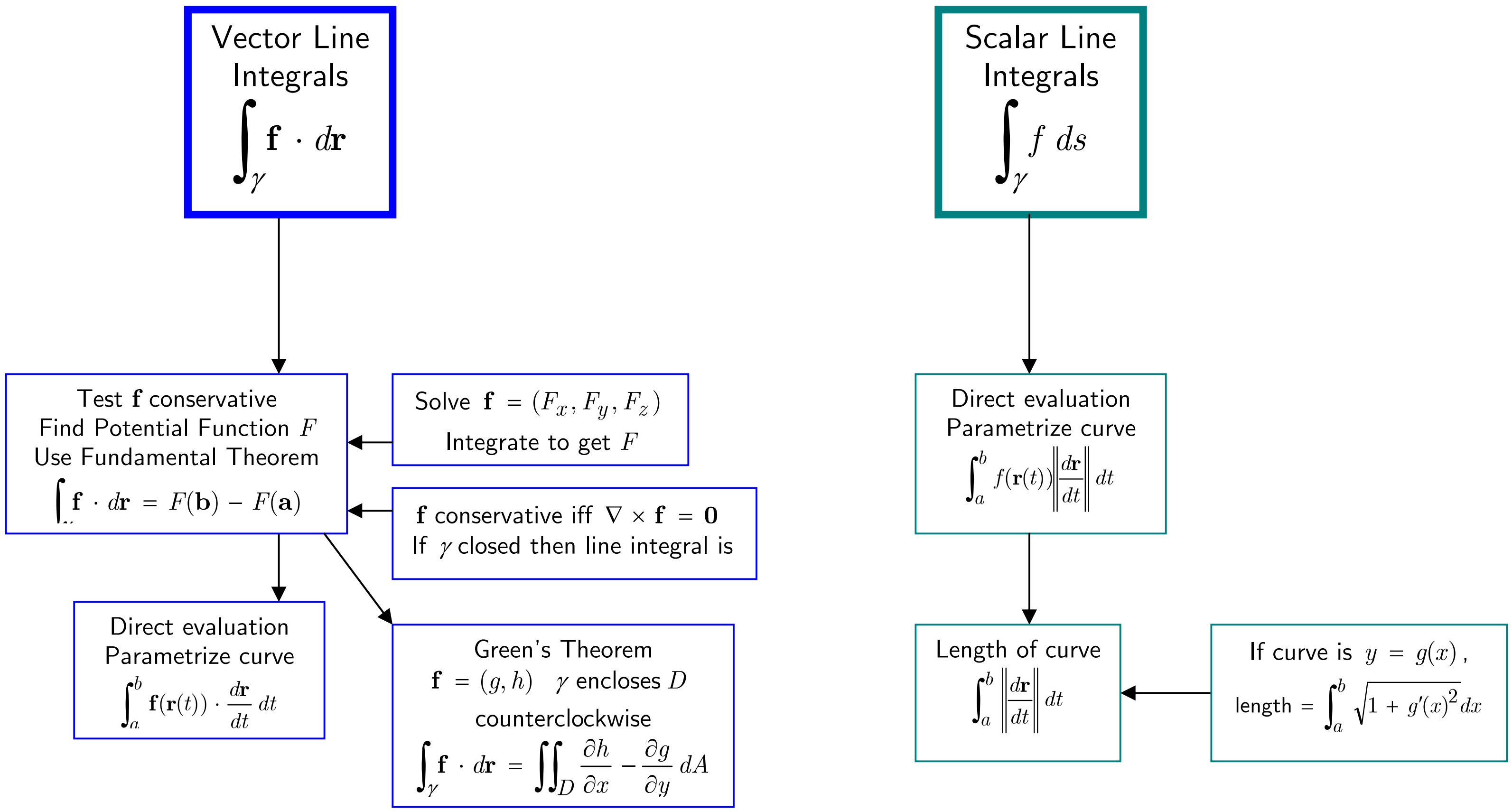
Summary



Vector Line Integrals

$$\int_\gamma \mathbf{f} \cdot d\mathbf{r}$$

Test $\mathbf{f}$ conservative
Find Potential Function F
Use Fundamental Theorem
$\int \mathbf{f} \cdot d\mathbf{r} = F(\mathbf{b}) - F(\mathbf{a})$

Solve $\mathbf{f} = (F_x, F_y, F_z)$
Integrate to get F

$\mathbf{f}$ conservative iff $\nabla \times \mathbf{f} = \mathbf{0}$
If γ closed then line integral is

Direct evaluation
Parametrize curve
$\int_a^b \mathbf{f}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt}\, dt$

Green's Theorem
$\mathbf{f} = (g, h)$ γ encloses D
counterclockwise
$\int_\gamma \mathbf{f} \cdot d\mathbf{r} = \iint_D \frac{\partial h}{\partial x} - \frac{\partial g}{\partial y}\, dA$

Scalar Line Integrals

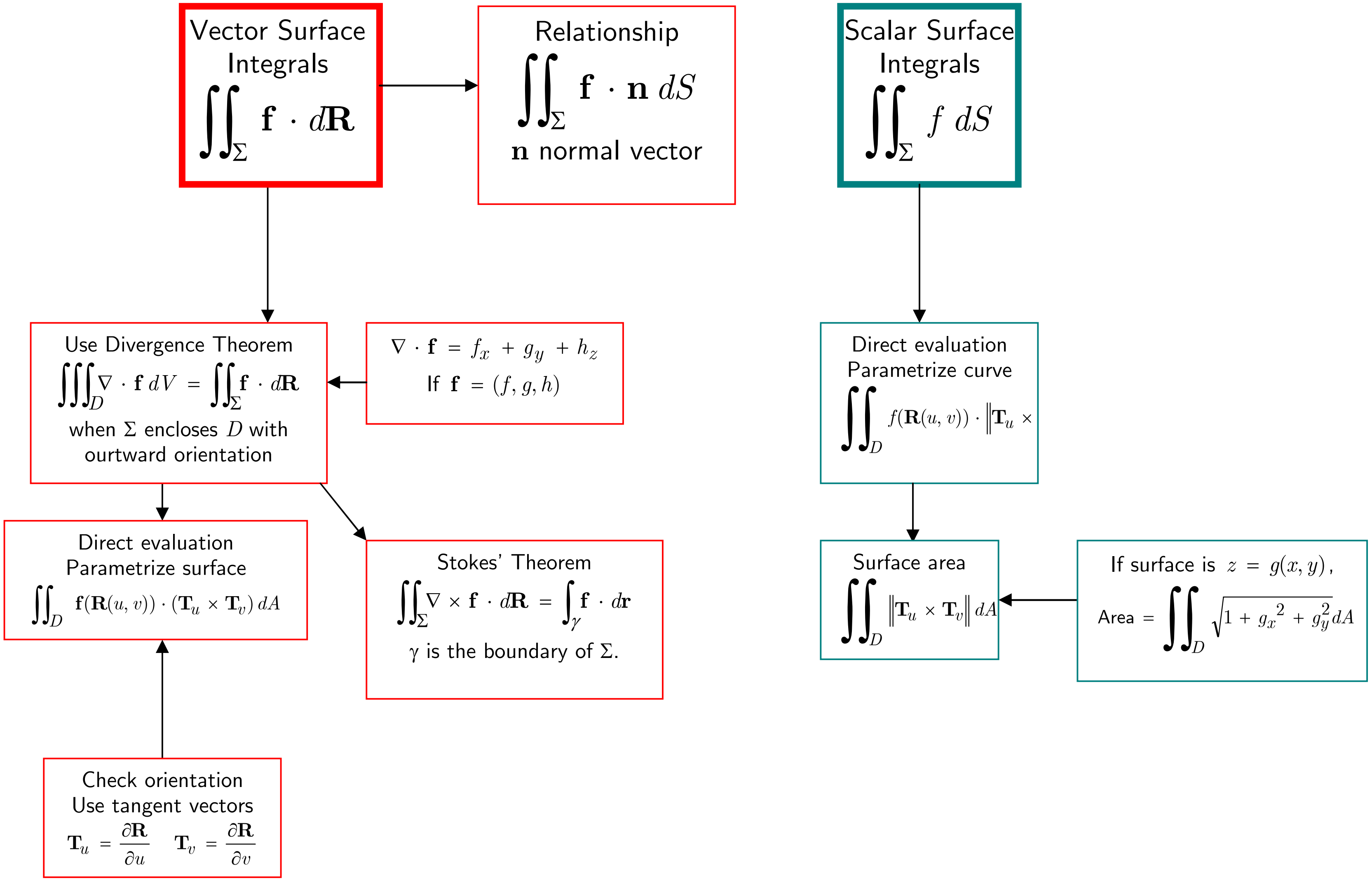
$$\int_\gamma f\, ds$$

Direct evaluation
Parametrize curve
$\int_a^b f(\mathbf{r}(t)) \left\| \frac{d\mathbf{r}}{dt} \right\| dt$

Length of curve
$\int_a^b \left\| \frac{d\mathbf{r}}{dt} \right\| dt$

If curve is $y = g(x)$,
length $= \int_a^b \sqrt{1 + g'(x)^2}\, dx$

Summary



Vector Surface Integrals
$$\iint_\Sigma \mathbf{f} \cdot d\mathbf{R}$$

Relationship
$$\iint_\Sigma \mathbf{f} \cdot \mathbf{n}\, dS$$
$\mathbf{n}$ normal vector

Use Divergence Theorem
$$\iiint_D \nabla \cdot \mathbf{f}\, dV = \iint_\Sigma \mathbf{f} \cdot d\mathbf{R}$$
when Σ encloses D with
ourtward orientation

$\nabla \cdot \mathbf{f} = f_x + g_y + h_z$
If $\mathbf{f} = (f, g, h)$

Direct evaluation
Parametrize surface
$$\iint_D \mathbf{f}(\mathbf{R}(u,v)) \cdot (\mathbf{T}_u \times \mathbf{T}_v)\, dA$$

Stokes' Theorem
$$\iint_\Sigma \nabla \times \mathbf{f} \cdot d\mathbf{R} = \int_\gamma \mathbf{f} \cdot d\mathbf{r}$$
γ is the boundary of Σ.

Check orientation
Use tangent vectors
$$\mathbf{T}_u = \frac{\partial \mathbf{R}}{\partial u} \quad \mathbf{T}_v = \frac{\partial \mathbf{R}}{\partial v}$$

Scalar Surface Integrals
$$\iint_\Sigma f\, dS$$

Direct evaluation
Parametrize curve
$$\iint_D f(\mathbf{R}(u,v)) \cdot \|\mathbf{T}_u \times$$

Surface area
$$\iint_D \|\mathbf{T}_u \times \mathbf{T}_v\|\, dA$$

If surface is $z = g(x,y)$,
$$\text{Area} = \iint_D \sqrt{1 + {g_x}^2 + g_y^2}\, dA$$

Line and surface integrals

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The purpose of these brief notes is to give an overview of line and surface integrals. The emphasis is on preparation for exam so there is a list of exercises for additional practice at the end. However, it is recommendation that you look at the Assignment 4 and 5 additional questions as well as the online lecture notes.

1. Line integrals

• A curve $\gamma \subset \mathbb{R}^n$ can be parametrized by representing it as $\mathbf{r}(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ for $a \leq t \leq b$. The curve starts at $\mathbf{r}(a)$ and end at $\mathbf{r}(b)$.

• A curve $\gamma \subset \mathbb{R}^2$ is often presented as $y = f(x)$ for $a \leq x \leq b$. In that case a natural parametrization is $\mathbf{r}(t) = (t, f(t))$ for $a \leq t \leq b$. In polar co-ordinates, a curve may often be given as $r = f(\theta)$ for $a \leq \theta \leq b$, in which case a good parametrization is $\mathbf{r}(t) = (f(t)\cos t, f(t)\sin t)$ for $a \leq t \leq b$.

• A curve which is piecewise smooth often should be parametrized in parts. The boundary of the unit square oriented counterclockwise could, for instance, be parametrized as $\mathbf{r}(t) = (t, 0)$ then $\mathbf{r}(t) = (1, t)$ then $\mathbf{r}(t) = (1 - t, 1)$ then $\mathbf{r}(t) = (0, 1 - t)$ for $0 \leq t \leq 1$.

• The line integral of a vector field $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ over curve γ parametrized by $\mathbf{r}(t)$ for $a \leq t \leq b$ is

$$\int_\gamma \mathbf{f} \cdot d\mathbf{r} = \int_a^b f(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt}\, dt.$$

The physical interpretation is the work done by force field f on a particle moving along γ. The work done is zero if f is perpendicular to γ at all points, for example, $f(x, y) = (x, y)$ and γ a circle.

• A vector field $f = (f_1, f_2, \ldots, f_n)$ with $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ is conservative if $f = \nabla F$ for some function $F : \mathbb{R}^n \to \mathbb{R}$. Here F is called a potential function. DarkBlueFundamental theorem for line integrals is that if $f = \nabla F$ then for any curve γ from a to b,

$$\int_\gamma \mathbf{f} \cdot d\mathbf{r} = F(b) - F(a).$$

In particular if γ is a closed curve then this integral is zero, and also the line integral is independent of the curve γ from a to b.

• We can find a potential function by solving the equations

$$F_{x_i} = f_i \quad \text{for} i = 1, 2, \ldots, n.$$

We integrate each equation with respect to x_i and remember that the constant of integration depends on all the other variables. However in most cases we encounter a potential function will be easy to find. For example, if $f(x, y, z) = (x, y, z)$ then a potential function is

$$F(x, y, z) = \frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{1}{2}z^2.$$

Now if γ is any curve say from $(0,0,0)$ to $(1,1,1)$, then

$$\int_\gamma f \cdot d\mathbf{r} = F(1,1,1) - F(0,0,0) = \frac{3}{2}.$$

This avoids the necessity to parametrize the curve γ.

- A test for conservativity is that $\nabla \times \mathbf{f}$ is zero when $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^3$. This is the curl of $\mathbf{f}$, sometimes denoted $\mathrm{curl}(f)$ and is exactly $(h_y - g_z, h_x - f_z, g_x - f_y)$. The notation $\nabla \times f$ suggests the formal determinant to remember the definition of the curl

$$\nabla \times \mathbf{f} = (h_y - g_z, h_x - f_z, g_x - f_y) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}.$$

Physically, the curl measures the angular velocity of a ball if the vector field is placed at a point in the vector field.

- The line integral of a scalar field $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}$ along an oriented curve γ is defined by

$$\int_\gamma f \, ds = \int_{t_a}^{t_b} f(\mathbf{r}(t)) \|r'(t)\| dt$$

where $\mathbf{r}(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ parametrizes γ from a to b, and

$$\|r'(t)\| = \sqrt{\left(\frac{dx_1}{dt}\right)^2 + \left(\frac{dx_2}{dt}\right)^2 + \cdots + \left(\frac{dx_n}{dt}\right)^2}.$$

- When $f = 1$, we get the arclength of the curve γ. In the special case that γ is $y = g(x)$ for $a \le x \le b$, we get the length of $y = g(x)$ from $x = a$ to $x = b$ is

$$\int_a^b \sqrt{1 + g'(x)^2} dx$$

since $\mathbf{r}(x) = (x, g(x))$ is the parametrization. We can relate line integrals over scalar and vector fields by

$$\int_\gamma \mathbf{f} \cdot d\mathbf{r} = \int_\gamma \mathbf{f} \cdot t ds$$

where $t = r'(t)/\|r'(t)\|$ is a unit tangent vector to γ.

- A few exercises on line integrals:
  - (a) Write the following curves in parametric form: (i) the circle $x^2 + y^2 = 1$ in the plane $z = 1$ (ii) the straight line from $(0,0,0)$ to $(1,2,3)$ (iii) the spiral $r = 1/\theta$ and $z = 1$ in cylindrical co-ordinates (iv) the parabola $y = z = x^2$ from $(0,0,0)$ to $(1,1,1)$.
  - (b) Find the line integrals of $\mathbf{f}(x,y,z) = (yz, xz, xy)$ over each of the curves in (a) where possible.
  - (c) Which of the vector fields are conservative (i) $\mathbf{f}(x,y) = (x,y)$ (ii) $\mathbf{f}(x,y,z) = (y,z,x)$ (iii) $\mathbf{f}(x,y,z) = (z \ln y, zx/y, x \ln y)$.
  - (d) Find the line integral of (c)(iii) over each curve in (a).
  - (e) Find the length of the curve $y = x^{4/3}$ for $0 \le x \le 1$.

2. Surface integrals

- Surfaces have two orientations since there are two normal vectors at each point. For instance a sphere can be oriented inwards or outwards. Given a unit normal vector at a point on a smooth surface, we know the orientation: for instance if we are told that $(1, 0, 0)$ is the unit normal to the unit sphere at $(1, 0, 0)$, then the sphere has outward orientation, otherwise the unit normal at that point is $(-1, 0, 0)$.

- An oriented surface $\Sigma \subset \mathbb{R}^n$ can be parametrized by representing it as

$$\mathbf{R}(u, v) = (x_1(u, v), x_2(u, v), \dots, x_n(u, v)) \quad \text{for } (u, v) \in D.$$

In the special case of a surface $z = f(x, y)$ for $(x, y) \in D$, it makes sense to use the parametrization $\mathbf{R}(u, v) = (u, v, f(u, v))$ for $(u, v) \in D$. In spherical co-ordinates, we might parametrize the unit sphere as $\mathbf{R}(u, v) = (\cos u \sin v, \sin u \sin v, \cos v)$ for $0 \leq u < 2\pi$ and $0 \leq v < \pi$.

- The only problem might be that the orientation is not correct when we parametrize a surface by $\mathbf{R}(u, v)$ for $(u, v) \in D$. The unit normal arising from $\mathbf{R}(u, v)$ is $(\mathbf{T}_u \times \mathbf{T}_v)/\|\mathbf{T}_u \times \mathbf{T}_v\|$ where

$$\mathbf{T}_u = \frac{d\mathbf{R}}{du} \quad \mathbf{T}_v = \frac{d\mathbf{R}}{dv}$$

are the tangent vectors in the u and v directions. If the orientation is not correct, then we use $R(v, u)$ as a parametrization instead of $\mathbf{R}(u, v)$. For example parametrization of the unit sphere above gives the unit sphere the inward orientation, since in that case at say at $(1, 0, 0)$ which corresponds to $(u, v) = (0, \pi/2)$, we get $\mathbf{T}_u \times \mathbf{T}_v = (-1, 0, 0)$.

- The surface integral of a vector field $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^3$ over an oriented surface Σ paramatrized by $\mathbf{R}(u, v) : (u, v) \in D$ is defined by

$$\iint_\Sigma \mathbf{f} \cdot d\mathbf{R} = \iint_D f(\mathbf{R}(u, v)) \cdot (\mathbf{T}_u \times \mathbf{T}_v) dW$$

It represents the flux of a fluid through Σ flowing according to the velocity field f.

- The surface integral of a scalar field $f : \mathbb{R}^3 \to \mathbb{R}$ over an oriented surface Σ is

$$\iint_\Sigma f dS = \iint_D f(\mathbf{R}(u, v)) \|\mathbf{T}_u \times \mathbf{T}_v\| dW$$

and in the case $f = 1$ we get the surface area of Σ. If the surface is $z = \phi(x, y)$ then by using parametrization $\mathbf{R}(u, v) = (u, v, \phi(u, v))$ we get the surface area formula

$$\iint_D \sqrt{1 + \phi_u^2 + \phi_v^2} dA$$

- We can relate the two surface integrals in a useful way by

$$\iint_\Sigma \mathbf{f} \cdot d\mathbf{R} = \iint_\Sigma \mathbf{f} \cdot \mathbf{n} dS$$

where $n = (\mathbf{T}_u \times \mathbf{T}_v)/\|\mathbf{T}_u \times \mathbf{T}_v\|$ is normal to Σ at each point. A particularly natural consequence is that f is tangent to Σ at each point of Σ, then the flux is zero, since $\mathbf{f} \cdot \mathbf{n} \equiv 0$.

- A few exercises on surface integrals:

(a) Parametrize the surfaces (i) $z = x^2 + y^2$, $x^2 + y^2 \leq 1$ (ii) $xy + (x+z)^2 = 1$, $0 \leq x \leq 1$, $0 \leq y \leq 1$ if the surfaces are oriented in the direction of $(0,1,0)$ at the point $(0,1,1)$.

(b) Determine the surface integrals of $\mathbf{f}(x,y,z) = (x,y,z)$ over each surface in (a).

(c) Prove that the surface area of a unit sphere is $4\pi^2$ using surface integrals.

(d) If Σ is the surface $z = \phi(x,y)$ and $\mathbf{f}(x,y,z) = (1,1,\phi_x + \phi_y)$, determine the flux of f across Σ.

3. Integral theorems

- The divergence of a vector field $\mathbf{f} = (f_1, f_2, \dots, f_n)$ is $\text{div}(\mathbf{f}) = \nabla \cdot \mathbf{f}$ and is defined by

$$f_{1x_1} + f_{2x_2} + \cdots + f_{nx_n}$$

so if $\mathbf{f}(x,y,z) = (f(x,y,z), g(x,y,z), h(x,y,z))$ then it is $\nabla \cdot \mathbf{f} = f_x + g_y + h_z$. Physically, divergence is a measure of how much the fluid expands per unit of volume. So the vector field $f(x,y) = (x,y)$ has divergence 2, whereas $f(x,y) = (y,x)$ has zero divergence.

- Some properties of divergence and curl are

$$\nabla \times \nabla f = 0 \quad \nabla \cdot (\nabla \times \mathbf{f}) = 0 \quad \nabla \cdot \nabla f = f_{x1x1} + f_{x2x2} + \cdots + f_{xnxn}$$

which can all be proved from the definitions.

- The divergence theorem (also known as Gauss' Theorem) says

$$\iiint_D \nabla \cdot f dV = \iint_\Sigma \mathbf{f} \cdot d\mathbf{R}$$

where D is a closed and bounded set in $\mathbb{R}^n$ with a smooth boundary Σ oriented outwards, and f is continuously differentiable on an open ball containing D. Physically, it makes sense: the flux of f across Σ equals the total divergence inside Σ.

- Consequences: we again see that if f is tangent to a closed surface Σ at every point, then we get

$$\iiint_D \nabla \cdot f dV = \iint_\Sigma \mathbf{f} \cdot d\mathbf{R} = \iint_\Sigma (\mathbf{f} \cdot \mathbf{n}) dS = 0$$

so the total divergence inside Σ is zero. Another consequence is that if there is no divergence inside Σ i.e. $\nabla \cdot f = 0$ then the flux of f through Σ is zero. Also, the flux of the curl of any vector field is zero since

$$\iint_\Sigma \nabla \times \mathbf{f} \cdot d\mathbf{R} = \iiint_D \nabla \cdot (\nabla \times f) dW = 0.$$

- An important special case of the divergence theorem is Green's Theorem. Green's Theorem says that if $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ is a vector field then for any closed curve γ with counterclockwise orientation enclosing a region D,

$$\int_\gamma \mathbf{f} \cdot d\mathbf{r} = \iint_D (h_x - g_y) dA$$

where $\mathbf{f} = (g, h)$. For example, if we want to find

$$\int_\gamma \mathbf{f} \cdot d\mathbf{r}$$

when $f(x, y) = (xy, x + y)$ and γ is the counterclockwise boundary of the square $[0, 1] \times [0, 1]$, we get

$$\int_0^1 \int_0^1 (1 - x) dx dy = \frac{1}{2}$$

much easier than working out the line integral.

- If $\Sigma \subset \mathbb{R}$ is an oriented surface, then the boundary γ of Σ inherits an orientation from Σ: walking along γ, the normal to Σ must be to the right. Stokes' Theorem states that if Σ is an oriented surface with smooth boundary γ, then

$$\iint_\Sigma (\nabla \times \mathbf{f}) \cdot d\mathbf{R} = \int_\gamma \mathbf{f} \cdot d\mathbf{r}.$$

Physically this says that the flux of the curl of a vector field through a surface Σ depends only on the boundary of Σ. Take, for instance, the surface Σ defined by $e^z(x^2 + z + y^2) = 1$ for $z \geq 0$ with outward orientation. This complicated surface has a simple boundary which is the circle $x^2 + y^2 = 1$. If $\mathbf{f}(x, y, z) = (x, y + z, z^3)$ then we conclude

$$\begin{aligned} \iint_\Sigma (\nabla \times \mathbf{f}) \cdot d\mathbf{R} &= \int_\gamma \mathbf{f} \cdot d\mathbf{r} \\ &= \int_0^{2\pi} f(\cos t, \sin t, 0) \cdot (-\sin t, \cos t, 0) dt \\ &= \int_0^{2\pi} (\cos t, \sin t, 0) \cdot (-\sin t, \cos t, 0) dt \quad = \quad 0. \end{aligned}$$

- Some exercises on integral theorems.

(a) Determine the surface integral $\iint_\Sigma \mathbf{f} \cdot d\mathbf{R}$ when Σ (with outward orientation) and $\mathbf{f}$ are (i) the surface of $[0, 1] \times [0, 1] \times [0, 1]$ and $\mathbf{f}(x, y, z) = (x + z^y, y + x/z, z + 1/x)$ (ii) the unit sphere and $\mathbf{f}(x, y, z) = (xy, xz, z^2)$

(b) Determine $\int_\gamma \mathbf{f} \cdot d\mathbf{r}$ when (i) γ is the unit circle and $f = (1 - x^2 - y^2, 1 - x^2 - y^2)$ (ii) γ consists of the unit circle oriented clockwise and the circle of radius $\frac{1}{2}$ oriented clockwise and $\mathbf{f} = (x^2 + y^3, y^2 - x^3)$.

(iii) Let h be any function which is 1 on a closed curve γ. Find $\iint_D (h_x - h_y) dA$ when D is the region enclosed by γ and D is (i) the unit square $[0, 1] \times [0, 1]$ (ii) the unit circle $x^2 + y^2 = 1$ (iii) the region between $y = 1 - x^2$ and $y = x^2 - 1$.

(c) Determine $\iint_\Sigma (\nabla \times \mathbf{f}) \cdot d\mathbf{R}$ where Σ is the paraboloid $z = 5 - x^2 - y^2$ for $z \geq 0$ with upward orientation and $\mathbf{f} = (z^2, -3xy, x^3y^3)$.

(d) Determine $\int_\gamma \mathbf{f} \cdot d\mathbf{r}$ where $\mathbf{f} = (z^2, y^2, x)$ and γ is the counterclockwise boundary of the surface $x + y + z = 1$ for $x \geq 0$ and $y \geq 0$ and $z \geq 0$.