ASSIGNMENT 1 : PROBABILISTIC COMBINATORICS jacques@ucsd.edu

Those question with a star next to them are considered more challenging – I strongly suggest you spend time on the easier ones first. Solutions to this assignment will be presented starting Monday October 22nd in class. In the assignment, $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ denote the set of natural numbers, integers and real numbers, respectively.

Question 1.

Prove the following inequalities, where $k, n, \alpha n \in \mathbb{N}$ and $n \geq k$ for (a) and (b). Do not use Stirling's Formula for (a) and (b). For (c), recall

$$H(\alpha) = -\alpha \log \alpha - (1-\alpha)\log(1-\alpha).$$

(a) $\displaystyle \frac{(n-k)^k}{k!} < \binom{n}{k} \leq \frac{n^k}{k!}$

(b) $\displaystyle \left(\frac{n}{k}\right)^k \leq \binom{n}{k} < \left(\frac{en}{k}\right)^k$

(c) $\displaystyle \binom{n}{\alpha n} < e^{nH(\alpha)}$

(d) $\displaystyle \binom{2n}{n} \sim \frac{4^n}{\sqrt{\pi n}}$ as $n \to \infty$.

Question 2.

(a)[1] Prove that for all $n, N \in \mathbb{N}$ with $N \geq n$,

$$R_2(n) > N - \binom{N}{n} 2^{1-\binom{n}{2}}.$$

(b) Deduce that $R_2(n) > e^{-1} n 2^{n/2}$ when n is large enough.

Question 3.

The Ramsey number $R_s(m,n)$ is the smallest number N such that in any red-blue colouring of $[N]^s$, the complete system of s-element subsets of $[N]$, there is a red $[m]^s$ or blue $[n]^s$.

(a) Prove that $R_s(m,n) \geq N$ whenever there exists $p \in [0,1]$ such that

$$p^{\binom{m}{s}} \binom{N}{m} + (1-p)^{\binom{n}{s}} \binom{N}{n} < 1$$

Does this give a polynomial or an exponential lower bound on $R_3(4,n)$ as a function of n?

(b)*[2] Prove that $R_3(4,n) \geq 2^{cn}$ for some constant $c > 0$.

---

[1] Hint : try deleting a vertex from each monochromatic copy of a complete graph K_n of order n appearing in a random red-blue colouring.

[2] Hint : Take a random orientation of the complete graph on n vertices, and then colour a triple red if it forms a directed triangle in the orientation, and blue otherwise.

Question 4.

Let Σ be a finite set. A word on Σ is an ordered list of symbols from Σ. A word is k-non-repetitive (k-nr) if no subword of length k appears at least twice. For each $k \in \mathbb{N}$,

(a) Prove that every k-non-repetitive word has length at most $|\Sigma|^k + k - 1$.

(b) Prove that a random word of length N is k-nr with positive probability if $N < (2|\Sigma|)^{k/2}$.

(c) Improve the bound in (b) to $N < (|\Sigma|^k - 1)/(4ek)$ using the local lemma.

(d)* Find a determistic algorithm for constructing a k-nr word of length $|\Sigma|^k + k - 1$.

Question 5*

(a) Prove that for each $n \in \mathbb{N}$, there exists a set $A \subset [n]$ of size $\lfloor n/2 \rfloor$ containing no arithmetic progression of length $\lceil 2 \log_2 n \rceil$ and such that no three consecutive integers in $[n]$ are absent from A.

(b) For each $\varepsilon > 0$, prove that there exists a constant $c(\varepsilon) > 0$ and a set $A \subset [n]$ of size at most $(1 + \varepsilon)\lfloor n/2 \rfloor$ such that A contains no arithmetic progression of length $\lceil c(\varepsilon) \log_2 n \rceil$ and no two consecutive integers in $[n]$ are absent from A.

Question 6.

The independence number of a graph G, denoted $\alpha(G)$, is the maximum size of a set of vertices with no edges between them.

(a) Write down the expected number of independent sets of m vertices in $G_{n,p}$.

(b) Show that for any $\varepsilon > 0$,

$$\mathbb{P}[\alpha(G_{n,p}) > (2 + \varepsilon) \log_{1/(1-p)} n] \to 0 \qquad \text{as } n \to \infty.$$

Question 7.

A hypergraph is a collection of subsets of a given set X. The transversal number $\tau(H)$ of a hypergraph H is the minimum cardinality of a set T of elements of X which meet every member of H (i.e. every set in H contains at least one element from T). Prove that if every element of H has size r (so H is a collection of r-element subsets of X), then for any $p \in [0, 1]$,

$$\tau(H) \leq p|X| + (1 - p)^r |H|.$$

Deduce that $\tau(H) \leq (|H| + |X| \log r)/r$.

Question 8.

Let $n \geq 2$ and let H be a hypergraph all of whose sets have size n. Prove that if H has 4^{n-1} edges, then there is a colouring of the points in H with four colours so that none of the sets in H is monochromatic (a set is monochromatic if all of its elements receive the same colour in the colouring).

Question 9.

Prove that there exists a constant $c > 1$ such that in $\mathbb{R}^d$ we can choose c^d distinct points such that no three of the points form an angle of at least ninety degrees.[3]

Question 10.

Let S be a set of binary strings of length $n \geq 2$. Then S is called universal if for each pair $\{i, j\} \subset \{1, 2, \dots, n\}$ and any binary string ab of length two, there is a string $s \in S$ such that $s_i = a$ and $s_j = b$.

(a) Prove that every universal set of strings of length n has size at least $\log_2 n$.

(b) Prove that there is a universal set of binary strings of size at most $8 \log n$.

Question 11*

The list chromatic number $\chi_\ell(G)$ of a graph G is the minimum number k such that if we assign to each vertex of the graph a list of colours of size k, it is possible to select a colour from each list to obtain a proper colouring of the graph. Show that every n-vertex bipartite graph G has $\chi_\ell(G) \leq \log_2 n + 1$. Show furthermore that there exist bipartite graphs G with $\chi_\ell(G) \gg \log_2 n$.

Question 12.

(a) Prove that the proportion of subsets of $[n]$ containing an arithmetic progression of length at least $c \log_2 n$ tends to zero if $c > 2$ and tends to one if $c < 2$, as $n \to \infty$.

(b) Construct a set $A \subset [n]$ of size $|A| \gg n^{1-1/k}$ containing no arithmetic progression of length $k + 1$ using the method of alterations.

(c) Let A be the set of all integers less than n whose digits in base $2k - 1$ are all less than k. Show that A does not contain an arithmetic progression of length $k + 1$ and $|A| \gg n^{\log_{2k-1} k}$.

Question 13.

Let H be any graph with $e > 1$ edges and v vertices. Prove that there exists an n-vertex graph G not containing H such that $e(G) \gg n^{2 - \frac{v-2}{e-1}}$.

Question 14.

Consider the equation integer equation $a_1 x_1 + a_2 x_2 + \cdots + a_k x_k = 0$ where $a_i \in \mathbb{Z} \backslash \{0\}$ and $x_i \in \mathbb{Z}$. Let

$$a = \sum \{a_i : a_i > 0\} \quad \text{and} \quad b = \sum \{-a_i : a_i < 0\}.$$

(a) Prove that if $a \neq b$, then there is a subset of $\mathbb{Z}_n$ of size $\lfloor \frac{n}{a+b} \rfloor$ with no solution $(x_1, x_2, \dots, x_k)$ to the given integer equation.

(b) Using the example on sum-free sets in the notes, prove that if $a \neq b$, then any set of positive integers of size n contains a subset of size $\lfloor \frac{n}{a+b} \rfloor$ with no solutions to the given integer equation.

---

[3] Hint : pick points of the d-dimensional hypercube independently with probability p, and then consider the expected number of triples forming a right-angled triangle.

Question 15.

A trivial solution to the Sidon equation $x_1 + x_2 = x_3 + x_4$ in an abelian group $\langle G, +\rangle$ is a solution (x_1, x_2, x_3, x_4) such that $\{x_1, x_2\} = \{x_3, x_4\}$.[4][5]

(a) Prove that every finite set $B \subset \mathbb{Z}$ contains a set A such that $|A| \gg |B|^{1/3}$ and A has only trivial solutions to the Sidon equation.

(b) Let $G = \mathbb{Z}_q$, where q is prime. Suppose p be a prime such that $q > 4p^2$ and let x_p denote the reduction of an integer x modulo p. Prove that the set

$$A = \{x + 4px_p^2 : 0 \leq x < p\}$$

has only trivial solutions to the Sidon equation in G.

(c)** Improve the result of (a) to $|A| \gg |B|^{1/2}$.

Question 16.

Let $n \in \mathbb{N}$. A maximal chain in $[n]$ is a sequence of subsets $(A_i)_{i=0}^{n}$ with $A_i \subset A_{i+1}$ and $|A_i| = i$. An antichain in $[n]$ is a family of sets where no set is contained in another.

(a) How many maximal chains in $[n]$ contain a fixed subset of $[n]$ of size r?

(b) Let $\mathcal{A}$ be an antichain of subsets of $[n]$. Show, by taking a randomly and uniformly chosen maximal chain in $[n]$, that

$$\sum_{r=0}^{n} \frac{|\mathcal{A}_r|}{\binom{n}{r}} \leq 1 \qquad \text{where } \mathcal{A}_r = \{A \in \mathcal{A} : |A| = r\}.$$

(c) Use (b) to show that $|\mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor}$.

Question 17.

A generalized triangle comprises three sets A_1, A_2, A_3 with $A_1 \cap A_2 \cap A_3 = \emptyset$ and with $A_1 \cap A_2, A_2 \cap A_3, A_3 \cap A_1$ all non-empty. The family of all s-element subsets of $[n]$ is denoted $[n]^s$.

(a) For $n \geq 5$, find a family $\mathcal{A} \subset [n]^3$ of size $\binom{n-1}{2}$ containing no generalized triangle.

(b) Let $\mathcal{A} \subset [n]^3$ contain no generalized triangle. For $x, y \in [n]$, let

$$\triangle(x, y) = |\{A \in \mathcal{A} : \{x, y\} \subset A\}|$$

and let X_{xy} be a Bernouilli random variable with success probability $\binom{n}{2}^{-1} \triangle(x, y)^{-1}$. For $A \in \mathcal{A}$ let $X_A = \sum_{\{x,y\} \subset A} X_{xy}$. Show that if $n \geq 5$, then

$$\mathbb{E}[X_A] \geq \frac{2}{(n-1)(n-2)}$$

(c) Let $Y = \sum_{A \in \mathcal{A}} X_A$. Show that $\mathbb{E}[Y] \leq 1$.

(d) Combine (b) and (c) to obtain $|\mathcal{A}| \leq \binom{n-1}{2}$.

---

[4] Hint : for part (a), take a random set, each element having probability p, and consider the expected number of non-trivial solutions to the given equation. Then alter the random set by deleting appropriate elements.

[5] Hint : for part (c), at least explain why the method we used for sum-free sets in class does not work.

Question 18.

Let A be a finite set of real numbers, and define

$$A + A = \{a + a' : a, a' \in A\} \quad \text{and} \quad A \cdot A = \{a \cdot a' : a, a' \in A\}.$$

(a) Using the appropriate results from the notes, prove that $|A + A||A \cdot A| \gg |A|^{5/2}$.

(b) Give an explicit example of a set A with $|A + A||A \cdot A| \asymp |A|^4$.

(c)* Give an example of a set $A \subset [|A|^2]$ with $|A + A||A \cdot A| \asymp |A|^4$. You may assume the unbelievably strong statement $\tau(i) \gg i/(\log i)^2$ for every even integer i, where $\tau(i)$ is the number of ways of writing i as a sum of two primes.

Question 19.

(a) Prove that $\mathbb{P}[X = 0] \leq \frac{\sigma^2}{\mu^2}$ when X has mean $\mu \neq 0$ and variance σ^2. Then show that the stronger inequality below is true:

$$\mathbb{P}[X = 0] \leq \frac{\sigma^2}{\mu^2 + \sigma^2}.$$

(b)* Let X and Y be discrete i.i.d real-valued random variables. Prove that

$$\mathbb{P}[|X - Y| \leq 2] \leq 3\mathbb{P}[|X - Y| \leq 1].$$

Show that the three cannot be replaced with a smaller number.

Question 20.

(a) Recall the definition of crossing number $\mathrm{cr}(G)$ of a graph G from the notes. A multigraph is a pair $G = (V, E)$ where V is a set of vertices and E is a multiset of pairs of elements of V (in other words, the same as a graph except we allow two vertices to be joined by many edges). Prove that every element of E appears at most k times in E, then

$$\mathrm{cr}(G) + k^2|V| \gg \frac{|E|^3}{k|V|^2}.$$

(b) Let P be a set of points in the plane. Let $d(P)$ denote the set of distinct distances between points of P, namely

$$\{d(u, v) : u, v \in P\}.$$

Prove that $d(P) \gg |P|^{2/3}$. You might wish to consider concentric circles around each $p \in P$ which cover all the other points of P, and consider the graph formed by shortest arcs between vertices of P of all of these circles. Then apply (a).

(c)* Use the Szemerédi-Trotter Theorem to improve the bound in (b) to $|P|^{4/5}$.