

**Question 1.**

- (a) Let  $X_1, X_2, \dots, X_n$  be independent random variables, and let  $X$  be their sum. Check that if  $\text{var}[X] = \sigma^2$ , then

$$\sigma^2 = \sum_{i=1}^n \text{var}[X_i].$$

- (b) Let  $Y$  be a random variable with  $|Y| \leq 1$  and  $\mathbb{E}(Y) = 0$ . Prove that for  $0 \leq t \leq 1$ ,

$$\mathbb{E}(e^{tY}) \leq 1 + t^2 \text{var}(Y).$$

- (c) Apply parts (a) and (b) to show that if  $X_1, X_2, \dots, X_n$  and  $X$  are as in (a) and  $|X_i| \leq 1$  and  $\mathbb{E}[X_i] = 0$  for all  $i$ , then for  $0 \leq t \leq 1$ ,

$$\mathbb{E}(e^{tX}) \leq e^{t^2 \sigma^2}.$$

- (d) Use Markov's Inequality to prove that

$$\mathbb{P}(|X| \geq \lambda \sigma) \leq 2e^{-\lambda^2/4}$$

where  $X$  is as in (c) and  $0 \leq \lambda \leq 2\sigma$ .

**Question 2.**

- (a) A random walk in the plane is walk on the lattice  $\mathbb{Z}^2 \times \mathbb{Z}^2$  in which the direction of each step is chosen uniformly from all four possible directions, independently for different steps. Prove that the probability that the random walk is at distance at least  $\lambda\sqrt{n}$  from the origin in  $n$  steps is at most  $8e^{-\lambda^2/4}$  for any positive real  $\lambda$ .
- (b) Determine the probability that the random walk returns to the origin infinitely often.

**Question 3.**

Let  $X = [n]$ , let  $\mathcal{S}$  be a family of subsets of  $X$ , and let  $\chi$  be a map  $X \rightarrow \{-1, 1\}$ . The discrepancy of  $\chi$  is

$$\text{disc}(\chi, \mathcal{S}) := \max \left\{ \left| \sum_{x \in E} \chi(x) \right| : E \in \mathcal{S} \right\}.$$

The discrepancy of  $\mathcal{S}$  is defined by  $\text{disc}(\mathcal{S}) := \min \{ \text{disc}(\chi, \mathcal{S}) : \chi \in \{-1, 1\}^X \}$ .

- (a) Let  $\mathcal{S}$  consist of all  $k$ -term progressions in  $X$ . Show  $\text{disc}(\mathcal{S}) \ll \sqrt{k \log n}$ .
- (b) Show that if  $|\mathcal{S}| = n$ , then  $\text{disc}(\mathcal{S}) \ll \sqrt{n \log n}$ .
- (c) Prove that there is a set system  $\mathcal{S}$  on  $X$  with  $|\mathcal{S}| = n$  and  $\text{disc}(\mathcal{S}) \gg \sqrt{n}$ .

**Question 4.**

Let  $k$  be a positive integer. A  $k$ -covering of  $K_n$  is a collection of bipartite subgraphs of  $K_n$  for which every edge of  $K_n$  appears in exactly  $k$  of the bipartite graphs. Let  $f_k(n)$  denote the smallest number of bipartite graphs in a  $k$ -covering of  $K_n$ .

- (a) Prove that  $f_1(n) = \lceil \log_2 n \rceil$  and  $f_k(n) \geq \lceil \log_2 n \rceil$ .
- (b) Show that  $f_k(n) \ll \log n + k$ , where the implied constant is absolute.
- (c) Let  $\phi(P)$  denote the largest angle formed by any three points of the set  $P \subset \mathbb{R}^2$ , where three collinear points are considered to form an angle of  $\pi$  radians. Let  $\phi(n)$  be the smallest value of  $\phi(P)$  over all  $n$ -point sets  $P \subset \mathbb{R}^2$ . Find  $\phi(n)$  for  $n \in \{3, 4, 5\}$ . Prove that

$$\pi - \phi(n) \sim \frac{\pi}{\log_2 n} \quad \text{as } n \rightarrow \infty.$$

[Hint : For a lower bound on  $\phi(n)$ , apply (a) as follows: let  $P$  be a set of  $n$  points and define  $H_r$  to be the graph of all pairs  $\{p, q\} \subset P$  spanning a line with gradient  $\tan \theta$ , where  $(r - 1)\pi \leq \theta n < r\pi$ . Show that each  $H_r$  is bipartite and that the  $H_r$  form a 1-covering of  $K_n$ . A hint for the upper bound is drawn below.]

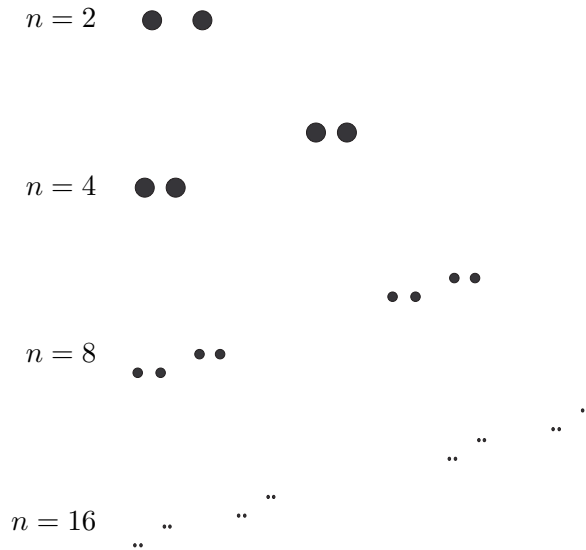


Figure 1 : Constructions with small largest angles for  $n \in \{2, 4, 8, 16\}$ .

**Question 5.**

Let  $\gamma(G)$  be the size of a smallest dominating set in a graph  $G$ . Find a sequence  $G_{n,d}$  of  $n$ -vertex graphs of minimum degree at least  $d$  containing in which  $\gamma(G_{n,d}) \gg \frac{n \log d}{d}$ .

**Question 6.**

Let  $X$  be a finite set,  $X^2 := X \times X$ . For a function  $f : X^2 \rightarrow \{-1, 1\}$ , define  $f_\Delta : X^3 \rightarrow \{-1, 1\}$  and  $f_\square : X^4 \rightarrow \{-1, 1\}$  by

$$\begin{aligned} f_\Delta(x_1, x_2, x_3) &= f(x_1, x_2)f(x_1, x_3)f(x_2, x_3) \\ f_\square(x_1, x_2, x_3, x_4) &= f(x_1, x_2)f(x_3, x_2)f(x_3, x_4)f(x_1, x_4). \end{aligned}$$

- (a) Find an  $f : X^2 \rightarrow \{-1, 1\}$  such that

$$\left| \sum_{x \in X^3} f_\Delta(x) \right| \in \{0, 1\}.$$

- (b) Prove that almost all functions  $f : X^2 \rightarrow \{-1, 1\}$  with  $|X| = n$  satisfy

$$\left| \sum_{x \in X^4} f_\square(x) \right| \ll n^3.$$

- (c) Find explicit functions  $f_n : [n] \rightarrow \{-1, 1\}$  satisfying the inequality in (b).

**Question 7.**

- (a) Let  $n$  and  $\Delta$  be positive integers. Design a probability space and events  $A_1, A_2, \dots, A_n$  therein such that  $\mathbb{P}(A_i) = \frac{1}{\Delta+1}$ , a dependency graph for the  $A_i$  has maximum degree  $\Delta$ , and

$$\mathbb{P}\left(\bigcap_{i=1}^n \bar{A}_i\right) = 0.$$

- (b) Prove the modified version of the local lemma, by slightly modifying the proof of the local lemma given in the notes.

**Question 8.**

- (a) Let  $G$  be a graph with no multiple edges, and let  $N(G)$  be the smallest positive integer  $N$  such that it is possible to associate an element  $x(v) \in \mathbb{Z}_N$  with vertex  $v \in V(G)$  so all elements of  $\{x(u) + x(v) \bmod N : \{u, v\} \in E(G)\}$  are distinct. Prove that  $N(G) \geq |E(G)|$  for every non-empty graph  $G$ . For each positive integer  $m$ , find a graph  $G$  with  $m$  edges such that  $N(G) = m$ .
- (b) Prove that for any graph  $G$  of maximum degree  $\Delta$ ,  $N(G) \ll \Delta^2|V(G)|$ . You may wish to randomly choose a number in  $\mathbb{Z}_N$  for each vertex, and apply the local lemma to the events  $A_{ef}$  that the ends of  $e, f \in E(G)$  have the same sum modulo  $N$ .

**Question 9.**

Let  $G$  be a  $d$ -regular graph where  $d \geq 4k^3$ . Prove that there is a partition  $(V_1, V_2, \dots, V_k)$  of  $V(G)$  such that for every vertex  $v \in V(G)$  and  $1 \leq i \leq k$ ,

$$\left| e(v, V_i) - \frac{d}{k} \right| < 4 \left( \frac{d \log d}{k} \right)^{\frac{1}{2}}$$

where  $e(v, V_i)$  is the number of edges of the form  $\{v, w\}$  with  $w \in V_i$ .

**Question 10.**

- (a) Let  $S$  be a finite set of integers. A vertex-distinguishing  $S$ -colouring of a graph  $G$  is a map  $\chi : E(G) \rightarrow S$  such that for every edge  $\{u, v\} \in E(G)$ ,

$$\sum_{e \ni u} \chi(e) \neq \sum_{f \ni v} \chi(f).$$

Prove, for all  $n \geq 3$ , that the complete graph  $K_n$  has a vertex-distinguishing  $S$ -colouring for some set  $S$  with  $|S| = 3$ .

- (b) Prove that there exists a finite set  $S$  such that if  $d$  is large enough, then every  $d$ -regular graph has a vertex-distinguishing  $S$ -colouring.

**Question 11.**

- (a) Let  $K_{n,n}$  denote the complete bipartite graph with  $n$  vertices in each part. Let  $\chi$  be an edge-colouring of  $K_{n,n}$  such that each colour appears at most  $\frac{1}{8}(n-4)$  times. Prove that for any  $\chi$ , there is a perfect matching in  $K_{n,n}$  all of whose edges have different colours.
- (b) Show that if  $n$  is large enough, then in any proper colouring of  $K_{n,n}$ , there is a perfect matching which does not contain three edges of the same colour. Deduce that in a latin square  $L$ , we can find a sequence of  $n$  entries, no two in the same row or column, such that no three of the entries are the same.

**Question 12.**

- (a) For an integer  $n$  with prime factorization  $n = p_1^{e_1} \dots p_k^{e_k}$ , let  $\Omega(n) = e_1 + e_2 + \dots + e_k$ . For a uniformly chosen  $x \in [n]$ , prove that  $\mathbb{E}(\Omega(x)) \sim \log \log n$ .
- (b) Prove that the variance of  $\Omega(x)$  is asymptotic to  $\log \log n$ , and show that

$$\frac{1}{n} |\{x \in [n] : |\Omega(x) - \log \log n| > \lambda \sqrt{\log \log n}\}| \rightarrow \frac{2}{\sqrt{\pi}} \int_{\lambda}^{\infty} e^{-t^2/2} dt.$$

- (c) Let  $\tau(n)$  denote the number of distinct divisors of an integer  $n$ . Using part (b) and the Erdős-Kac Theorem, prove that  $(\log 2)(\log \log n)$  is a normal order for  $\log \tau(n)$ .