Question 1.

Let $d \in \mathbb{N}$, $c \in \mathbb{R}^+$, and let G_n be an n-vertex d-regular graph. Suppose that vertices of G_n are selected independently with probability p, where $p \sim c(dn/2)^{-1/2}$ as $n \to \infty$. Let X_n be the number of edges between selected vertices of G_n. Prove that

$$\mathbb{P}(X_n = 0) \to e^{-\frac{1}{2}c^2} \qquad \text{as } n \to \infty.$$

Question 2.

(a) Let $p_c(\mathbb{Z}^d)$ denote the critical threshold for percolation in the d-dimensional integer lattice. Prove that $0 < p_c(\mathbb{Z}_d) < 1$.

(b) Find a cubic lattice $\mathbb{L}$ – a lattice where every element is immediately comparable to exactly three others – such that $p_c(\mathbb{L}) = 0$.

Question 3.

(a) Let H denote a random k-uniform hypergraph on $[n]$, in which hyperedges appear uniformly and independently. Let X denote the number of isolated vertices of H. Prove that a sharp threshold function for the event $X = 0$ is given by

$$\tau(n) = \frac{\log n}{\binom{n-1}{k-1}}.$$

(b) Determine a threshold $\tau(n)$ for a k-term arithmetic progression in a subset of $[n]$ whose elements are chosen uniformly and independently. Determine asymptotically the probability of a k-term progression when elements are chosen with probability $c \cdot \tau(n)$, where $c \in \mathbb{R}^+$.

Question 4.

For a fixed graph H, let X_H denote the number of **induced** copies of H in $G_{n,p}$. Determine a function $\tau = \tau(n)$ and $\upsilon = \upsilon(n)$ such that

$$\mathbb{P}(X_H = 0) \to \begin{cases} 0 & \text{if } \frac{p}{\tau} \to 0 \text{ or } \frac{1-p}{\upsilon} \to 0 \\ 1 & \text{if } \frac{p}{\tau} \to \infty \quad \text{and} \quad \frac{1-p}{\upsilon} \to \infty. \end{cases}$$

You might wish to use Suen's Inequality.

Question 5.

(a) Let G be a graph whose vertices are independently infected at time zero with probability p. Suppose that any vertex v becomes infected if at least half its neighbours are infected. Let $A = A_G$ be the event that the entire graph becomes infected in finite time. Show that if G has n vertices and minimum degree at least $4 \log n$, then $\mathbb{P}(A)$ tends to zero or one according as $p \leq \frac{1}{2}$ or $p > \frac{1}{2}$.

(b) If G is the $n \times n$ box, show that no configuration of at most $n-1$ infected sites is in A. Then prove that there are constants $a, b > 0$ such that

$$\mathbb{P}(A) \to \begin{cases} 0 & \text{if } p \leq \frac{a \log n}{n} \\ 1 & \text{if } p \geq \frac{b \log n}{n} \end{cases}$$

(c) If G is the wheel on $n+1$ vertices, find a real number p^* such that

$$\mathbb{P}(A) \to \begin{cases} 1 & \text{if } \ p > p^* \\ p & \text{if } \ p \leq p^* \end{cases}$$

Question 6.

(a) Prove that if $Z : \Omega \to [0,1]$ is a random variable and $f : [0,1] \to \mathbb{R}$ is a convex function, then $\mathbb{E}(f(Z)) \leq \mathbb{E}(Z)f(1) + (1 - \mathbb{E}(Z))f(0)$.

(b) Let $X_1, X_2, \ldots, X_n$ be negatively correlated random variables with expectation μ and range $[N]$, and $S = \sum_{i=1}^{n} X_i$. Prove that

$$\mathbb{P}(S - \mathbb{E}(S) \geq \lambda) \leq e^{-\frac{\lambda^2}{2\mu n N}}.$$

Question 7.

(a) Let $\epsilon \in \mathbb{R}^+$. Let X denote the number of triangles in $G_{n,p}$ and let $\mu = p^3\binom{n}{3}$. Suppose that $p^2 n \to 0$. Prove that for some constant $a > 0$,

$$\mathbb{P}(X \leq (1-\epsilon)\mu) \leq e^{-a\epsilon^2 p^3 n^3}.$$

(b) Prove that for some constant $b > 0$,

$$\mathbb{P}(X \geq (1+\epsilon)\mu) \leq e^{-b\epsilon^2 p^3 n^2}.$$

Question 8.

(a) Let $(X_i)_{i\in\mathbb{N}}$ be a martingale with difference sequence $(Y_i)_{i\in\mathbb{N}}$ and $Y_1 = X_1$. Then $(X_i)_{i\in\mathbb{N}}$ is c-Lipschitz with exceptional probability η if for all $i \in \mathbb{N}$ where $i > 1$,

$$\mathbb{P}(|Y_i| > c) \leq \eta.$$

Prove that for $\lambda \geq 0$,

$$\mathbb{P}(|X_n - \mathbb{E}(X_n)| > \lambda) \leq 2e^{-\lambda^2/2\sum c_i^2} + 2n\eta.$$

(b) Let G be a graph of maximum degree $\triangle$. Colour the vertices of G independently with a uniformly chosen colour from $[k]$ where $k \in \mathbb{N}$. Uncolour any two adjacent vertices of the same colour. Show that if $\triangle > \log n$ and X is the number of vertices which retain their colour, then

$$\mathbb{P}(|X - \mathbb{E}(X)| > \omega(n)(n \log n)^{1/2}) \to 0$$

for any function $\omega(n) \to \infty$.

Question 9.

(a) Let $X_1, X_2, \ldots, X_n$ be random variables where $X_i : \Omega \to \mathbb{R}$, and let Y_i be measurable with respect to the σ-field F_i generated by $X_1, X_2, \ldots, X_i$. Suppose that for some constants $a_i, c_i \in \mathbb{R}$,

$$\mathbb{E}(Y_i - Y_{i-1} \mid F_{i-1}) < a_i \quad \text{and} \quad |Y_i - Y_{i-1} - a_i| < c_i \quad \text{a.s.}$$

Let $A := \sum_{i=1}^{n} a_i$. Show that for $\lambda > 0$,

$$\mathbb{P}(\max_{i\in[n]} Y_i > Y_0 + A + \lambda) < \exp\Big(-\frac{\lambda^2}{2\sum c_i^2}\Big).$$

(b) Let G_0 be the empty graph on n vertices, and let G_i be formed from G_{i-1} for $i \in \mathbb{N}$ by adding an edge between a uniformly chosen pair of non-adjacent vertices of degree at most two, if such a pair exists, otherwise let $G_i = G_{i-1}$. Prove that for any $\varepsilon > 0$, if $t \geq (1+\varepsilon)n$, then G_t a.a.s contains a component with a linear number of vertices as $n \to \infty$.

Question 10.

(a) Let $X : \Omega \to \mathbb{R}^+$ be an $f(s) = ds$ certifiable k-Lipschitz random variable, where $\Omega = \prod_{i=1}^r \Omega_i$. Prove that

$$|\mathbb{E}(X) - \mathbb{M}(X)| \leq k\gamma\sqrt{d\mathbb{E}(X)}$$

for some constant γ, where $\mathbb{M}(X)$ is a median of X. You might recall $\mathbb{E}(X) \geq \frac{1}{2}\mathbb{M}(X)$ and apply Talagrand to each of the quantities

$$\mathbb{P}(|X - \mathbb{M}(X)| > ik\sqrt{d\mathbb{M}(X)}) \quad \text{for } i \geq 0.$$

(b) Let $k \geq 2$, and let H be a k-uniform hypergraph. Let H' be a random subgraph of H consisting of edges of H chosen independently with probability p. Prove that $|\bigcup\{E : E \in H'\}|$ is highly concentrated at its expectation when $|H| \to \infty$.

Question 11.

(a) Throughout this question, G is a triangle-free graph of maximum degree $\triangle$. Let $\chi(G)$ denote the chromatic number of G. Prove that if there is a proper $(\triangle+1-r)$-colouring of a subset of vertices of G such that at least r colours appear at least twice in the neighbourhood of every vertex of G, then $\chi(G) \leq \triangle + 1 - r$.

(b) Prove that G is contained in a $\triangle$-regular triangle-free graph.

(c) Consider the following colouring procedure on a $\triangle$-regular graph: assign to each vertex independently and uniformly a colour from $[d]$ where $d = \lfloor \triangle/2 \rfloor$, and remove the colour on any vertex which has the same colour as one of its neighbours. Let X_v denote the number of colours appearing at least twice in the neighbourhood of v. Prove that $\mathbb{E}(X_v) \geq \triangle/e^6 - 1$.

(d) Use Talagrand's Inequality to show that

$$\mathbb{P}(|X_v - \mathbb{E}(X_v)| > (\log \triangle)\sqrt{\mathbb{E}(X_v)} < \frac{1}{4\triangle^5}$$

(e) Use the local lemma to prove that if $\triangle(G) = \triangle$ is large enough, then with positive probability, none of the events $X_v < \triangle/2e^6 + 1$ occur. Deduce from (a) and (b) that $\chi(G) \leq (1 - 1/2e^6)\triangle$ whenever G is a triangle-free graph of maximum degree $\triangle$.

Question 12.

Let $B_t(n)$ denote a random $n \times n$ bipartite graph formed by adding edges t times to the empty $n \times n$ bipartite graph t times, where a new edge is chosen uniformly from the set of pairs of vertices which are not already edges. Prove that the random variables $M = \min\{t : B_t(n) \text{ has a perfect matching}\}$ and $\min\{t : B_t(n) \text{ has no isolated vertices}\}$ are a.a.s equal, and determine an asymptotic formula for M as a function of n.

Question 13.

(a) Let V be an n-element set and k an integer such that kn is even. Let f be a uniformly chosen pairing of $V \times [k]$, and let $G(k)$ denote the multigraph on V obtained by contracting all vertices $(v, i) : 1 \leq i \leq k$ to a single vertex $v \in V$. Let A be the event that $G(k)$ has no multiple edges and no loops. Using the Brun Sieve, show that for each fixed k,

$$\mathbb{P}(A) \to e^{-\frac{1}{4}(1-k^2)}$$

where $f(k)$ is a linear function of k.

(b) Deduce from (a) the asymptotic number of k-regular simple graphs (no multiple edges or loops allowed) on n vertices.