1 - Introduction to hypergraphs

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1 Introduction

In this course you will learn broad combinatorial methods for addressing some of the main problems in extremal combinatorics and then in other areas of mathematics. This includes the probabilistic method, spectral and polynomial methods, and methods from higher algebra.

We begin with an introduction to hypergraphs, which gives a taste of different representations of hypergraphs, linear hypergraphs, and Turán-type problems, including existence of Turán densities and classification of zero Turán densities. Thereafter we delve deeper into some of the classical theorems of hypergraph theory, including various theorems on intersecting families such as Sperner’s Theorem, the LYM Inequality, the Erdős-Ko-Rado Theorem, Hilton-Milner Theorem, Deza-Frankl Theorem, Erdős-Rado Theorem and Frankl-Wilson Theorem. The tools include linear algebraic methods, polynomial methods, the delta-system method, compressions and shadows. The general Turán problem is considered in the framework of analytic methods and Lagrangians, and we consider some specific case studies where the exact answers are known.

We focus on extremal graph theory, where we present Turán’s Theorem, the Erdős-Stone Theorem, the Kővari-Sós-Turán Theorem, the Even Cycle Theorem and the Erdős-Gallai Theorem. We introduce the method of dependent random choice and we consider particular algebraic constructions for the bipartite Turán problems. The Combinatorial Nullstellensatz is introduced, and we study the problem of existence of $k$-regular subgraphs in dense graphs.
In the last part of the course we consider probabilistic and semirandom methods, which include the Rödl Method in various forms with many applications. A number of probabilistic tools which are useful specifically in extremal combinatorics is introduced.

1.1 Asymptotic notation

Let \( f, g : \mathbb{N} \to \mathbb{R}^+ \) be functions. Then we use the following notation:

\[
\begin{align*}
    f &= O(g) & \text{For some constant } c > 0, f(n) \leq cg(n) \text{ for all } n \in \mathbb{N}. \\
    f &= \Omega(g) & g = O(f) \\
    f &= o(g) & \lim_{n \to \infty} \frac{f(n)}{g(n)} = 0. \\
    f &= \omega(g) & g = o(f). \\
    f &= \Theta(g) & f = O(g) \text{ and } f = \Omega(g). \\
    f &\sim g & \lim_{n \to \infty} \frac{f(n)}{g(n)} = 1. \\
    f &\lesssim g & \limsup_{n \to \infty} \frac{f(n)}{g(n)} \leq 1.
\end{align*}
\]

If \( f = \Theta(g) \), then we say that \( f \) and \( g \) have the same order of magnitude.

1.2 Real number inequalities

We frequently use Jensen’s Inequality for convex functions: if \( f : \mathbb{R} \to \mathbb{R} \) is convex and \( x_1, x_2, \ldots, x_n \) are real numbers and \( x \) is their average, then

\[
f(x_1) + f(x_2) + \cdots + f(x_n) \geq nf(x).
\]

This will often be stated as by convexity. We also recall Taylor series for standard functions such as \( e^x \) and \( \log x \), and in particular we have the useful inequalities: for \( 0 \leq x < 1 \), \( \log(1 - x) \leq -x \) and for \( x > -1 \), \( \log(1 + x) \leq x \). These lead to the inequality

\[
\prod_{i=1}^{n}(1 + x_i) \leq \exp(\sum_{i=1}^{n} x_i)
\]

for reals \( x_1, x_2, \ldots, x_n > -1 \). We also recall for \( x \geq -1 \) and \( y \geq 0 \),

\[
(1 + x)^y \geq 1 + xy
\]
and more generally if \( x = x_n \) and \( y = y_n \) and \( x_n y_n \sim c \), then \((1 + x)^y \sim e^c\). A particularly useful formula to remember is Stirling’s Formula \( n! \sim \sqrt{2\pi n} \cdot (n/e)^n \), and more specifically,

\[
n! = e^{\theta_n} \sqrt{2\pi n} (n/e)^n
\]

where \( \frac{1}{12n+1} \leq \theta_n \leq \frac{1}{12n} \). For non-negative reals \( x_i, y_i \), the Cauchy-Schwarz inequality is used in the form

\[
\left( \sum_{i=1}^{n} x_i y_i \right)^2 \leq \left( \sum_{i=1}^{n} x_i^2 \right) \left( \sum_{j=1}^{n} y_j^2 \right).
\]

Hölder’s Inequality, which generalizes Cauchy-Schwarz, states that if \( \frac{1}{p} + \frac{1}{q} = 1 \) where \( p, q \geq 1 \), then

\[
\sum_{i=1}^{n} x_i y_i \leq \left( \sum_{i=1}^{n} x_i^p \right)^{1/p} \cdot \left( \sum_{i=1}^{n} y_i^q \right)^{1/q}.
\]

Any further inequalities required will be developed locally.

### 1.3 Basic definitions and hypergraph notation

Let \( 2^V \) denote the power set of a set \( V \) and let \( \binom{V}{k} \) denote the subsets of \( V \) of size \( k \). A hypergraph is a pair \((V, E)\) where \( V \) is a set and \( E \subseteq 2^V \). The elements of \( V \) are called vertices and the elements of \( E \) are called edges. A multihypergraph is a pair \((V, E)\) where \( V \) is a set and \( E \) is a multiset of subsets of \( V \) – in other words we allow repeated edges. If \( H \) is a hypergraph, we write \( V(H) \) for the set of vertices and \( E(H) \) for the set of edges. If \( E \subseteq \binom{V}{k} \) then \( H \) is called \( k \)-uniform and \( H \) is called a \( k \)-graph. We often write only \( E \) for a hypergraph \((V, E)\) with the understanding that \( V(H) = \bigcup_{e \in E} e \) and write \( e(H) \) instead of \(|E(H)|\). The density of a \( k \)-graph \( H \) on \( n \) vertices is \( e(H)/\binom{n}{k} \). For a set \( S \subset V(H) \), let

\[
N_H(S) = \bigcup_{e \in H, e \ni S} e \setminus S
\]

denote the neighborhood of \( S \) and

\[
d_H(S) = |\{e \in E(H) : e \ni S\}|
\]

denote the degree of \( S \). When there is no ambiguity we suppress the subscript \( H \). Also note that for 2-graphs \( H, d_H(v) = |N_H(v)| \) for every vertex \( v \in V(H) \). A hypergraph \( H \) is \( d \)-regular if for every \( v \in V(H) \), \( d_H(v) = d \). An isolated vertex is a vertex of degree zero.
Proposition 1. If $H$ is a hypergraph, then
\[
\sum_{e \in E(H)} \binom{|e|}{r} = \sum_{S \subseteq V(H) : |S| = r} d_H(S).
\]

In particular, if $H$ is an $n$-vertex $d$-regular $k$-graph, then $e(H) = dn/k$. Proposition 1 generalizes the handshaking lemma for graphs, which is the case $k = 2$ and $r = 1$.

It is convenient to let $K^t_k$ be the complete $k$-graph on $t$-vertices (the edge set is $\binom{V_k}{k}$ with $|V| = t$. Let $K_{t,k}$ denote the complete $k$-partite $k$-graph with parts of size $t$, namely, $V(K_{t,k}) = V_1 \sqcup V_2 \sqcup \cdots \sqcup V_k$ where $|V_1| = |V_2| = \cdots = |V_k|$ and $E(K_{t,k}) = \{v_1, v_2, \ldots, v_k : v_1 \in V_1, v_2 \in V_2, \ldots, v_k \in V_k\}$. For $k = 2$ and $k = 3$ we may sometimes write $K_{t,t}$ and $K_{t,t,t}$ for the complete $k$-partite $k$-graph.

If $S$ is a set of vertices in a hypergraph $H$, then we denote by $H - S$ the hypergraph with vertex set $V(H) \setminus S$ and edge set $\{e \in H : e \cap S = \emptyset\}$. If $E$ is a set of edges in $H$, let $H - E$ denote the hypergraph with vertex set $V(H)$ and edge set $E(H) \setminus E$. If $S \subseteq V(H)$, then $H[S]$ is the hypergraph $(S, \{e \in H : e \subseteq S\})$: this is the subgraph induced by $S$.

The shadow hypergraph of a hypergraph $H$, denote $\partial H$, is the hypergraph with vertex set $V(H)$ and edge set $\{e - \{x\} : e \in E(H), x \in e\}$. We denote by $\partial^k H$ the $k$th shadow hypergraph, namely, $\partial(\partial(\partial(\partial H) \ldots))$. In many instances, one can obtain information on $H$ from $\partial H$, and many classical theorems on intersections in hypergraphs are naturally proved using shadows. The link hypergraph of a set $S \subset V(H)$ is the hypergraph $H_S$ with vertex set $V(H) \setminus S$ and edge set $\{e \setminus S : e \in E(H), S \subset e\}$.

1.4 Representations of hypergraphs

In the last section we defined a hypergraph to be a pair $(V, E)$ where $E$ is a family of subsets of $E$. There are other useful representations of hypergraphs, each of which we visit briefly below.

Inclusion representations. The bigraph representation of a hypergraph $H = (V, E)$ is a bipartite graph with parts $V$ and $E$, where $v \in V$ is joined to $e \in E$ if $v \in E$. This allows one sometimes to use graph theory on the bigraph representation to deduce facts about the original hypergraph. In general, one can consider any
inclusion representation, whereby we create a bipartite graph with parts \((\mathcal{V}_r)\) and \(E\) and where \(S \in (\mathcal{V}_r)\) is joined to \(e \in E\) if \(S \subseteq e\). It is sometimes convenient to consider the inclusion matrix of the hypergraph, which is exactly the incidence matrix of the inclusion representation. In other words, if \(H\) is a hypergraph, we can form the matrix \(I\) whose rows are indexed by \((\mathcal{V}_r)\) and whose columns are indexed by \(E\), where \(I_{S,e} = 1\) if \(e \subseteq S\) and \(I_{S,e} = 0\) otherwise. This is sometimes useful to bring the tools of linear algebra on \(I_{S,e}\) to give information about \(H\).

**Tensor representation.** In graph theory, this is often done via the adjacency matrix of the graph, and the natural generalization of this to \(k\)-graphs is the adjacency tensor. Namely, given a \(k\)-graph \(H = (V, E)\), form the tensor \(A\) indexed by \(V \times V \times \cdots \times V\) where \(A_{v_1, v_2, \ldots, v_k} = 1\) if \(\{v_1, v_2, \ldots, v_k\} \in E(H)\) and \(A_{v_1, v_2, \ldots, v_k} = 0\) otherwise. It is possible to introduce some linear algebra here via multilinear forms, but the theory is rather sparse, and there does not seem to be a useful notion of rank and spectrum of tensors for combinatorial purposes.

**Graphs from hypergraphs.** If \(H\) is a \(k\)-graph, then we could pick numbers \(r, s\) such that \(r+s = k\), and we could form a graph whose vertex set is \((\mathcal{V}_r) \cup (\mathcal{V}_s)\) consisting of edges \(\{R, S\}\) where \(R \in (\mathcal{V}_r)\) and \(S \in (\mathcal{V}_s)\) and \(R \cup S = e\) for some \(e \in E(H)\). When \(r \neq s\), this graph is bipartite, but when \(r = s\), this is not necessarily bipartite.

A well-known graph is the Kneser graph \(K_{n,k}\): given \(n\) and \(k\), the vertex set of \(K_{n,k}\) is \((\mathcal{V}_k)\) where \(|\mathcal{V}| = n\), and the edge set of \(K_{n,k}\) is the set of disjoint pairs of elements of \((\mathcal{V}_k)\). In other words, two \(k\)-element subsets of \(V\) are adjacent if they are disjoint. Then a \(k\)-graph \(H = (V, E)\) can be viewed as a subgraph of \(K_{n,k}\). If, for instance, every two edges of \(H\) intersect, then \(H\) corresponds to an independent set in \(K_{n,k}\). This is one of the key connections in the proof of the Erdős-Ko-Rado Theorem.

**Duality.** Finally, we mention a natural notion of duality in hypergraphs. Given a multihypergraph \(H = (V, E)\), we can form a dual multihypergraph \(H^* = (E, V^*)\) where the elements of \(V^*\) are indexed by the vertices \(v \in V\) and where the edge \(e_v\) indexed by \(v\) is precisely \(\{e \in E : v \in e\}\). For instance, if \(H\) is a \(d\)-regular \(k\)-uniform hypergraph, then \(H^*\) is a \(k\)-regular \(d\)-uniform multihypergraph. In the event that \(H\) is a 2-regular hypergraph, \(H^*\) is a multigraph, and we can bring in the tools of graph theory.

**Other representations.** In applications the hypergraphs that arise have are sometimes constructed from algebraic, geometric or group theoretic sources. For instance,
given an abelian group $\Gamma$ and a set $S \subset \Gamma$, one may define the notion of a Cayley $k$-graph $H$ with vertex set $\Gamma$ and edge set $\{\{x_1, x_2, x_3, \ldots, x_k\} : x_1 + x_2 + \cdots + x_k \in S\}$. Another example is to take as vertices the 1-dimensional subspaces of an $n$-dimensional vector space over $\mathbb{F}_q$, and the edges as the sets of 1-dimensional subspaces which form $k$-dimensional subspaces. This leads to the classical construction of projective planes when $n = 3$ and $k = 2$. For a geometric example, one could take as vertex set the toroidal grid $\mathbb{T}_n = \mathbb{Z}_n \times \mathbb{Z}_n$, and as edge set all triples of the form $\{(x, y), (x + a, y), (x, y + b)\} \subset B$ where $a \geq 0$ and $b \geq 0$ and addition is modulo $n$. These kinds of constructions will come up fairly frequently in extremal problems, and are often worthy of study in their own right.

1.5 Turán-type problems for hypergraphs

A hypergraph $F$ is a subgraph of hypergraph $H$ if $V(F) \subseteq V(H)$ and $E(F) \subseteq E(H)$, and we write $F \subseteq H$. If $\mathcal{F}$ is a set of hypergraphs, we say that $H$ is $\mathcal{F}$-free if $F \not\subseteq H$ for all $F \in \mathcal{F}$. The central problem of extremal combinatorics is to determine or estimate

$$ex(n, \mathcal{F}) := \max\{e(H) : H \subset 2^{[n]} \text{ and } H \text{ is } \mathcal{F}\text{-free}\}.$$

We also let

$$ex_k(n, \mathcal{F}) = \max\{e(H) : H \subset \binom{[n]}{k} \text{ and } H \text{ is } \mathcal{F}\text{-free}\}.$$

These problems are collectively referred to as Turán-type problems. In this course we focus mainly on the second problem. Any $\mathcal{F}$-free $k$-graph with $n$ vertices and $ex_k(n, \mathcal{F})$ edges is called an extremal hypergraph for $\mathcal{F}$.

**Proposition 2.** For any $\mathcal{F}$, the following limit exists

$$\lim_{n \to \infty} \frac{ex_k(n, \mathcal{F})}{\binom{n}{k}}.$$

**Proof.** The existence of the limit follows from the fact that $\pi_n(\mathcal{F}) = ex_k(n, \mathcal{F})/\binom{n}{k}$ is non-increasing as a function of $n$ and bounded below by 0. To see this, count pairs $(e, T)$ where $T$ is a set of $n - 1$ vertices in an $\mathcal{F}$-free $n$-vertex hypergraph $H$, and $e \subset T$ is an edge of $H$. The number of pairs is at most $n \cdot ex_k(n - 1, \mathcal{F})$. On the other hand, the number of pairs is exactly

$$\sum_{e \in E(H)} \binom{n - k}{n - 1 - k} = e(H)(n - k).$$
Therefore
\[
\frac{\text{ex}_k(n-1), \mathcal{F}}{\binom{n-1}{k}} \geq e(H) \frac{(n-k)}{n(n-1)} = \frac{e(H)}{\binom{n}{k}}.
\]
Taking limits we get
\[
\pi_{n-1}(\mathcal{F}) \geq \pi_n(\mathcal{F}).
\]

The limit is denoted \( \pi(\mathcal{F}) \), the Turán density of \( \mathcal{F} \). The above proof has a number of powerful consequences. The first is the Simonovits supersaturation theorem:

**Proposition 3.** Fix \( k \geq 2 \) and let \( F \) be a \( t \)-vertex \( k \)-graph. For every \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that if \( H \) is any \( n \)-vertex \( k \)-graph with \( \text{ex}_k(n, F) + \varepsilon n^k \) edges, then \( H \) contains \( \delta \binom{n}{t} \) copies of \( F \).

This proposition says that the copies of \( F \) saturate the moment a \( k \)-graph passes above the density \( \pi(F) \). A consequence of the proof of Proposition 2 is that

\[
\pi_n(K^k_t) \leq \pi_t(K^k_t) = 1 - \left( \frac{t}{k} \right)^{-1},
\]

and therefore:

**Corollary 4.** For all \( n \),

\[
\pi(K^k_t) \leq 1 - \left( \frac{t}{k} \right)^{-1}.
\]

In general, this corollary is not tight. An analytic approach of Sidorenko using Feynman integrals greatly improves the upper bound; we shall come to this topic later. In the case of graphs, we shall see that one can determine \( \pi(F) \) exactly for any \( F \), via the Erdős-Stone Theorem. For hypergraphs, very little is known. Let us consider a case study. Simple bounds for \( \pi(K^3_4) \) are as follows:

**Proposition 5.** \( \frac{5}{9} \leq \pi(K^3_4) \leq \frac{1}{\sqrt{2}} \).

**Proof.** First we give the upper bound. Let \( H \) be an \( n \)-vertex \( K^3_4 \)-free hypergraph. We observe that since any four vertices in the hypergraph contain at most three edges.

\[
\sum_{\substack{S \subseteq V(H) \atop |S| = 2}} \left( \frac{d(S)}{2} \right) \leq 3\binom{n}{4}
\]

By convexity, and Proposition 1, the sum on the left is at least

\[
N \cdot \left( \frac{3e(H)/N}{2} \right)
\]
where \( N = \binom{n}{2} \). A calculation gives \( e(H) \lesssim \frac{1}{\sqrt{2}} \binom{n}{3} \), as required.

For the lower bound, a construction of a dense \( K_4^3 \)-free hypergraph is required. Form an \( n \)-vertex hypergraph \( H = (V, E) \) where \( V = X \cup Y \cup Z \) and \( |X| \leq |Y| \leq |Z| \leq |X| + 1 \), and where \( E \) consists of all edges \( \{x, y, z\} \) with \( x \in X \), \( y \in Y \) and \( z \in Z \), and all edges \( \{a, b, c\} \) where \( a, b \in X \) and \( c \in Y \), or \( a, b \in Y \) and \( c \in Z \), or \( a, b \in Z \) and \( c \in X \). The total number of edges is

\[
|X||Y||Z| + \binom{|X|}{2}|Y| + \binom{|Y|}{2}|Z| + \binom{|Z|}{2}|X| \approx \frac{5n^3}{54}.
\]

Dividing by \( \binom{n}{3} \) gives \( \pi(K_4^3) \geq \frac{5}{9} \). \( \square \)

Equality holds in the upper bound of Proposition 5 only if almost all quadruples carry exactly three triples. However, for any \( x, y \in V(H) \), there are \( \binom{n-d(x,y)-2}{2} \) quadruples containing \( S \) which carry at most two triples. This idea allows the bound in Proposition 5 to be improved substantially.

We do not investigate the best bounds on \( \pi(K_t^k) \) at this stage, except to say that the state of the art is to considering larger subsets of \( V(H) \), one uses the method of flag algebras to generate inequalities which give bounds on \( \pi(K_4^3) \) which are much close to \( \frac{5}{9} \). However, Turán’s conjecture remains open:

**Conjecture 6.** \( \pi(K_4^3) = \frac{5}{9} \).

One difficulty with Turán’s conjecture is that there are infinitely many asymptotically extremal non-isomorphic hypergraphs, as constructed by Kostochka, other than the ones given in the proof above. The situation for \( K_t^k \) is apparently even more difficult, and Erdős offers one thousand dollars for the determination of \( \pi(K_t^k) \) for any pair \((t, k)\) with \( t > k > 3 \).

### 1.6 Degenerate Turán Problems

In preceding sections, we saw that determining \( \pi(\mathcal{F}) \) for general families of hypergraphs \( \mathcal{F} \) is an open question. However, a special case is to determine those families \( \mathcal{F} \) for which \( \pi(\mathcal{F}) = 0 \). These are called Degenerate Turán Problems. An interesting feature of these problems is that the extremal hypergraphs tend to be random-like (in a sense that will be made precise), and the constructions tend to have very rich geometric or algebraic structure.
1.6.1 \( k \)-partite \( k \)-graphs

We aim to show that \( k \)-partite \( k \)-graphs are those \( k \)-graphs responsible for a family \( \mathcal{F} \) having zero Turán density. A \( k \)-graph \( H \) is said to be \( k \)-partite if \( V(H) \) admits a partition \( V_1 \sqcup V_2 \sqcup \cdots \sqcup V_k \) such that for every \( e \in E(H) \) and every \( i \leq k \), \( |e \cap V_i| = 1 \). The sets \( V_i \) are called the parts of \( H \). If \( H \) is any \( k \)-graph, then a cut of \( H \) is a \( k \)-partite subgraph containing all vertices of \( H \). A key useful result due to Erdős and Kleitman shows that every \( k \)-graph has a relatively dense \( k \)-partite subgraph:

**Proposition 7.** If \( H \) is a \( k \)-graph, then \( H \) has cut with at least \( \frac{k!}{k^k} e(H) \) edges.

**Proof.** We randomly, uniformly and independently assign a color \( c(v) \) from \( \{1, 2, \ldots, k\} \) to each vertex \( v \) of \( H \). In other words, for \( 1 \leq i \leq k \), \( \mathbb{P}(c(v) = i) = 1/k \). Then fixing an edge \( e \in E(H) \), the probability that all the vertices in \( e \) have different colors (we say \( e \) is multicolored) is \( \frac{k!}{k^k} \). In particular, the expected number of multicolored edges of \( H \) is

\[
\frac{k!}{k^k} e(H)
\]

by linearity of expectation. Pick an instance of a coloring of \( V(H) \) such that at least \( \frac{k!e(H)}{k^k} \) edges of \( H \) are multicolored, and let \( V_i \) be the set of vertices that received color \( i \) for \( i \leq k \). Then the \( k \)-graph \( H' \) of all multicolored edges is the required cut. \( \square \)

If we would like the parts in the cut to have as close to the same size as possible, then a slightly different proof can be used. A partition \( V_1 \sqcup V_2 \sqcup \cdots \sqcup V_k \) of a set \( V \) is called balanced if \( |V_1| \leq |V_2| \leq \cdots \leq |V_k| \leq |V_1| + 1 \) – in other words, the parts are as equal in size as possible. A balanced cut of a \( k \)-graph \( H \) is a cut whose parts form a balanced partition of \( V(H) \). A hint of the proof of the following is given.

**Proposition 8.** If \( H \) is a \( k \)-graph, then \( H \) has a balanced cut with at least \( \frac{k!}{k^k} e(H) \) edges.

**Proof.** We count in two ways pairs \((e, P)\) such that \( P = V_1 \sqcup V_2 \sqcup \cdots \sqcup V_k \) is a balanced partition of \( V(H) \) and \( |e \cap V_i| = 1 \) for \( 1 \leq i \leq k \). The details are left as an exercise. \( \square \)

We leave the reader to think about an algorithm which efficiently finds, given a \( k \)-graph \( H \), a balanced cut with at least \( \frac{k!}{k^k} e(H) \) edges.
1.6.2 Kövari-Sós-Turán Theorem for graphs

We return now to the zero Turán density question. To start gently, we first consider the case of graphs: for which families $F$ of graphs is $\pi(F) = 0$? It turns out that the bipartite graphs are responsible for this. To prove this, we use the Kövari-Sós-Turán Theorem:

**Theorem 9.** For all $t \geq s \geq 2$,
\[
\operatorname{ex}_2(n, K_{s,t}) \leq (t - 1)^{1/s} n^{2-1/s} + (s - 1)n.
\]

**Proof.** Let $H$ be a $K_{s,t}$-free graph with $n$ vertices. We claim
\[
\sum_{v \in V(H)} \binom{d_H(v)}{s} \leq (t - 1) \binom{n}{s}.
\]

To see this, note that the left hand side counts pairs $(v, S)$ where $v \in V(H)$ and $S \subset N_H(v)$ has size $s$. If there are more than $(t - 1) \binom{n}{s}$ such pairs, then some fixed set $S \subset V(H)$ of size $s$ must have appeared in more than $t - 1$ pairs $(v, S)$. In other words, there must be vertices $v_1, v_2, \ldots, v_t$ which all have $S$ in their neighborhood. This is precisely a copy of $K_{s,t}$ with parts $S$ and $T = \{v_1, v_2, \ldots, v_t\}$. By convexity and Proposition 1,
\[
\sum_{v \in V(H)} \binom{d_H(v)}{s} \geq n \binom{d}{s}
\]
where $d = 2e(H)/n$ is the average degree of $H$. Therefore
\[
n \binom{d}{s} \leq (t - 1) \binom{n}{s}
\]
and this shows
\[
\frac{n}{s!} (d - s + 1)^s \leq (t - 1) \frac{n^s}{s!}.
\]
This implies
\[
d \leq (t - 1)^{1/s} n^{1-1/s} + s - 1
\]
and since $d = 2e(H)/n$ we find
\[
e(H) \leq (t - 1)^{1/s} n^{2-1/s} + (s - 1)n.
\]
Therefore $\pi_n(K_{s,t}) = O(n^{-1/s})$ and taking limits $\pi(K_{s,t}) = 0$ as required. \qed
From this it quickly follows that $\pi(F) = 0$ if and only if $F$ contains a bipartite graph.

**Corollary 10.** A family $F$ of graphs has $\pi(F) = 0$ if and only if $F$ contains a bipartite graph.

**Proof.** If $F$ contains no bipartite graph, then the complete bipartite graphs $K_{a,n-a}$ with parts of size $a = \lfloor n/2 \rfloor$ and $n-a$ are $F$-free and asymptotically have density $\frac{1}{2}$. Therefore $\pi(F) \geq \frac{1}{2}$ in this case. If $F$ contains a bipartite graph $F$, then $F$ is contained in $K_{s,t}$ for some $s,t$. By Theorem 9,

$$\text{ex}_2(n,F) \leq \text{ex}_2(n,K_{s,t}) \leq (t-1)^{1/s}n^{2-1/s} + (s-1)n.$$ 

Therefore $\pi(F) = 0$. \qed

In fact from Theorem 9 we deduce that for any finite family $F$ containing a bipartite graph, there exists $\gamma > 0$ such that $\text{ex}_2(n,F) = O(n^{2-\gamma})$, so the density zero result is in a very strong sense. One may ask for the behavior of $\text{ex}_2(n,F)$ for other bipartite graphs, however this is known as the bipartite Turán problem and is generally very difficult. Erdős and Simonovits made a number of related conjectures, perhaps the most important being the exponent conjecture:

**Conjecture 11.** For every finite family $F$ containing a bipartite graph $F$, there exists an exponent $\alpha \in [1,2)$ such that

$$\text{ex}_2(n,F) = \Theta(n^{\alpha}).$$

This conjecture is wide open for most classes of graphs, and is not known for such simple graphs as octagons, cubes and $K_{4,4}$. We will return in depth to the exponent question when we introduce more advanced counting methods and constructions. We shall also see that there are infinite families of graphs with no exponent. In the case of $k$-graphs with $k > 2$, the situation is even more complicated. We shall see that Conjecture 11 cannot be extended to $k$-graphs: there are finite families $F$ of $k$-graphs with no exponent.

### 1.6.3 $k$-partite $k$-graphs have zero Turán density

We claimed that if $F$ is a family of $k$-graphs then $\pi(F) = 0$ if and only if $F$ contains a $k$-partite $k$-graph. A key proposition is the following:
Proposition 12. Let \( d \geq 1 \) be a real number and \( k > r \geq 1 \). If \( H \) is an \( n \)-vertex \( k \)-graph with \( (d-1)(\begin{pmatrix} n \\ r \end{pmatrix}) \) edges, then \( H \) has a subgraph \( H' \) such that the degree of every \( r \)-set in \( H' \) is at least \( d \).

We leave the proof as an exercise. We return now to show \( \pi(F) = 0 \) when \( F \) contains a \( k \)-partite \( k \)-graph. It is sufficient to show \( \pi(K_{t,k-1}) = 0 \) for every \( t \in \mathbb{N} \).

Proceed by induction on \( k \), noting that when \( k = 2 \) this follows from the Kövari-Sós-Turán Theorem. Suppose \( \pi(K_{t,k-1}) = 0 \) and let \( H \) be an \( n \)-vertex \( k \)-graph with \( \varepsilon n^{k-1} \) edges. By Proposition 12, \( H \) has a subgraph \( J \) with minimum degree at least \( \varepsilon n^{k-1} \). Suppose \( |V(J)| = m \). Then each link hypergraph \( J_v \) is a \((k-1)\)-graph with at least \( \varepsilon n^{k-1} \geq \varepsilon m^{k-1} \) edges. Since \( \pi(K_{t,k-1}) = 0 \), the number of copies of \( K_{t,k-1} \) in \( J_v \) is at least \( \delta m^{(k-1)t+1} \).

This shows that there is some set \( X \) of \((k-1)t \) vertices such that there are at least \( \delta m \) pairs \((v,K)\) where \( V(K) = X \). There are \( T = \frac{t(t(k-1))}{t,t,t,\ldots,t} \) ways to partition \( X \) into sets of size \( t \), and so there are \( \delta m/T \) pairs \((v,K)\) where \( K \) is some fixed \( K_{t,k-1} \). For large enough \( m \), \( \delta m/T \geq t \), which means we get \( t \) identical copies of \( K_{t,k-1} \) which are in the link graphs \( J_{v_1}, J_{v_2}, \ldots, J_{v_t} \) for some vertices \( v_1, v_2, \ldots, v_t \in V(H) \). This implies \( H \) contains all edges of the form \( \{v_1\} \cup e \) where \( e \in K_{t,k-1} \), which is precisely \( K_{t,k} \) on the vertex set \( \{v_1, v_2, \ldots, v_t\} \cup V(K_{t,k-1}) \).

In fact, this counting proof can be substantially refined to give an upper bound on \( \text{ex}_k(n, K_{t,k}) \) of the form \( O(n^{k-1/t^{k-1}}) \), akin to the Kövari-Sós-Turán Theorem when \( k = 2 \).

1.6.4 Kövari-Sós-Turán Theorem for hypergraphs

The Kövari-Sós-Turán Theorem allows us to actually count copies of \( K_{s,t} \) in a dense graph. The following proposition is due to Erdős and Moon:

Proposition 13. Let \( G \) be an \( n \)-vertex graph of density \( p \) such that

\[
e(G) = p \binom{n}{2} \geq \frac{1}{2} s^{1+1/s} n^{2-1/s} + 2sn.
\]

Then the number of copies of \( K_{s,s} \) in \( G \) is at least \( \Omega(p^{s^2} n^{2s}) \).
Proof. First we note that if
\[ M = \sum_{v \in V(G)} \binom{d_G(v)}{s} \]
then using the lower bound on \( e(G) \) and convexity,
\[ M \geq n \binom{2e(G)/n}{s} \geq sn^s. \]

Now the number of copies of \( K_{s,s} \) in a graph \( G \) is exactly
\[ \frac{1}{2} \sum_{S \subseteq V(G)} \binom{f(S)}{s} \]
where the sum is over sets \( S \) of size \( s \), and \( f(S) \) is the number of vertices which are adjacent to every vertex in \( S \). Since \( M \geq sn^s \geq s(n_s)^s \),
\[ \sum_{S \subseteq V(G)} \binom{f(S)}{s} \geq \binom{n}{s} \cdot \binom{M}{(n_s)} = \Omega(n^{s-s^2} M^s). \]

Since \( M = \Omega(e(G)^s n^{1-s}) \), \( n^{s-s^2} M^s = \Omega(p^{s^2} n^{2s}) \), as required. \( \square \)

We now use this to show that the families of 3-graphs \( \mathcal{F} \) such that \( \pi(\mathcal{F}) = 0 \) are precisely the 3-partite 3-graphs. This follows from the following theorem.

**Theorem 14.** For any \( t \geq 2 \),
\[ \text{ex}_k(n, K_{t,k}) = O(n^{k-1/t^{k-1}}). \]

Proof. We prove the result for \( k = 3 \) and leave the generalization to \( k > 3 \) as an exercise. Let \( H \) be a \( K_{t,t,t} \)-free \( n \)-vertex 3-graph. Suppose \( e(H) = \omega(n^{3-1/t^2}) \). By Proposition 12, there is a subgraph \( J \) of \( H \) of minimum degree \( d \geq e(H)/n \). Let \( m \) be the number of vertices of \( J \). The number of edges in the link graph \( J_v \) is exactly the degree of \( v \) in \( J \), so we conclude \( e(J_v) \geq e(H)/n = \omega(m^{2-1/t^2}) \) for every \( v \in V(J) \). By Proposition 13, \( J_v \) contains \( \omega(m^{2t-1}) \) copies of \( K_{t,t} \). We therefore have \( \omega(m^{2t}) \) pairs \( (v, K) \) where \( v \in V(J) \) and \( K = K_{t,t} \subset J_v \). It follows that for some fixed \( K = K_{t,t} \), there are \( \omega(1) \) pairs \( (v, K) \), and in particular, there are vertices \( v_1, v_2, \ldots, v_t \in V(J) \) such that \( K \subset J_{v_i} \) for all \( i \leq t \). Then \( \{v_1, v_2, \ldots, v_t\} \cup V(K) \) is the vertex set of \( K_{t,t,t} \subset J \subset H \), contradicting that \( H \) is \( K_{t,t,t} \)-free. Therefore \( e(H) = O(n^{3-1/t^2}) \). \( \square \)