1 Introduction

If $\mathcal{F}$ is a family of graphs containing a bipartite graph, then for some $\delta > 0$, $\text{ex}(n, \mathcal{F}) < n^{2-\delta}$ by the Kővari-Sós-Turán Theorem. Erdős and Simonovits [5] conjectured that for every finite family graph $\mathcal{F}$ of graphs, there exists $\alpha = \alpha(\mathcal{F})$ such that $\text{ex}(n, \mathcal{F}) = \Theta(n^\alpha)$. This $\alpha$ is called the exponent of $\mathcal{F}$. If $F$ is a bipartite graph containing a cycle, then in general it is not known if $\mathcal{F} = \{F\}$ has an exponent, even for simple graphs such as $F = K_{4,4}$ and $F = C_8$. In these notes, we give an approach to finding upper bounds on the exponent using a method called Dependent Random Choice, which has many other applications besides. We refer the reader to Fox and Sudakov [6] for a survey of dependent random choice.

2 Dependent random choice

The method of dependent random choice was developed from early ideas of Gowers [7] and Kostochka and Rödl [8]. If $G = (V, E)$ is an $n$-vertex graph with average degree $d$, then on average a sequence $R \in V^r$ has $d^r/n^{r-1}$ common neighbors. However, there may be many sequences $R$ which have no common neighbors, for instance if $G$ has a small dense component plus many isolated vertices. The idea of dependent random choice is to locate a large set $Z \subset V$ such that every element of $Z^r$ has many common neighbors, by letting $Z$ be the common neighborhood of a random sequence of vertices. This has the effect of biasing the elements of $Z^r$ to have many common neighbors as oppose to a randomly chosen set $Z$ of the same size.
The following notation will be used. Throughout, $a, b, m, r, s, t, n$ are positive integers. Let $G = (V, E)$ be a graph and $R \in V^r$ i.e. $R$ is a sequence of $r$ vertices of $V$ with replacement. Then $N(R) = \bigcap_{v \in R} N(v)$ and $d(R) = |N(R)|$. We say $R$ is $b$-rich if $d(R) \geq b$ and $b$-poor otherwise. A set $Z \subset V$ is $(r, b)$-rich if every sequence in $Z^r$ is $b$-rich. The next theorem gives a density condition on a graph which guarantees a large $(r, b)$-rich subset of vertices:

**Theorem 1.** Let $G = (V, E)$ be an $n$-vertex graph of average degree $d$, where

$$d^n n^{1-a} - (b-1)^a n^{r-a} > m - 1.$$  

Then there exists $Z \in \binom{V}{m}$ such that $Z^r$ is $(r, b)$-rich.

**Proof.** Randomly select a sequence $S \in V^a$, and let $W$ be the number of $b$-poor sequences in $N(S)^r$. Let $R$ be the set of $b$-poor sequences in $V^r$. Then

$$\mathbb{E}(W) = \frac{1}{n^a} \sum_{R \in R} d(R)^a \leq (b-1)^a n^{r-a}.$$

Delete one vertex from each $b$-poor sequence in $N(S)^r$ to get a set $Z \subset N(S)$. Then by convexity,

$$\mathbb{E}(|Z|) \geq \frac{1}{n^a} \sum_{S \in V^a} d(S) - (b-1)^a n^{r-a}$$

$$= \frac{1}{n^a} \sum_{v \in V} d(v)^a - (b-1)^a n^{r-a}$$

$$\geq d^n n^{1-a} - (b-1)^a n^{r-a} > m - 1.$$  

Therefore there exists a choice of $S$ so that $|Z| \geq m$.  

2.1 Embedding bipartite graphs

Theorem 1 allows the embedding of bipartite graphs $F$ with parts $X$ and $Y$ of size $s$ and $t$, where all vertices in $Y$ have degree at most $r$. We use the following elementary embedding lemma:

**Lemma 2.** Let $F$ be a bipartite graph with parts $X$ and $Y$ such that $|X| = s$ and $|Y| = t$ and every vertex of $Y$ has degree at most $r$. If $G = (V, E)$ is a graph and $Z \in \binom{V}{s}$ is $(r, t+s-r)$-rich, then $F \subseteq G$.  

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Proof. Bijectively map the vertices of $F$ in $X$ to the vertices of $Z$. Now suppose we have embedded vertices $y_1, y_2, \ldots, y_u$ of $Y$ in $V(G) \setminus Z$, where $u < t$. Then $y_{u+1}$ has at most $r$ neighbors amongst $y_1, y_2, \ldots, y_u$, which we may assume are $y_1, y_2, \ldots, y_q$. Then the sequence $(y_1, y_2, \ldots, y_r)$ is $(t+s-r)$-rich, so \{${y_1, y_2, \ldots, y_r}$\} has at least $t$ common neighbors in $V(G) \setminus Z$. Since $u < t$, there exists a vertex $v \in V(G) \setminus Z$ adjacent to $y_1, y_2, \ldots, y_r$, and now we may map $y_{u+1}$ to $v$. 

**Theorem 3.** Let $F$ be a bipartite graph with parts $X$ and $Y$ such that $|X| = s$ and $|Y| = t$ and every vertex of $Y$ has degree at most $r$. Then for any integer $a \geq r$,$$
abla ex(n, F) \leq \frac{1}{2}(s-1)\frac{a}{n}n^{2-\frac{1}{a}} + \frac{1}{2}(t+s-r-1)n^{1+\frac{a-1}{a}}.

In particular, $ex(n, F) = O(n^{2-\frac{1}{r}})$.

Proof. If $G$ is an $n$-vertex $F$-free graph with average degree $d$, then by the embedding lemma and Theorem 1 with $m = s$ and $b = t + s - r$, $$d^a n^{1-a} - (t + s - r - 1)^a n^{r-a} \leq s - 1.$$

This gives the bound on $ex(n, F)$. Taking $a = r$ we obtain $ex(n, F) = O(n^{2-\frac{1}{r}})$.

2.2 Subdivisions

A subdivision of a graph $G$, denoted $S(G)$ is a graph obtained by replacing each edge with a path of length two with the same ends as the edge, such that all the paths are internally disjoint. For a graph $n$ vertices and $m$ edges, the subdivision is a bipartite graph with parts of size $m$ and $n$, and every vertex in the part of size $m$ has degree two. The following is proved using Theorem 3 with a careful choice of $a$.

**Theorem 4.** If $G$ has $n$ vertices and $pn^2$ edges then $S(K_{\lceil p\sqrt{m} \rceil}) \subset G$.

Proof. Let $s = \lceil p\sqrt{m} \rceil$. By Theorem 3 with $r = 2$, a calculation gives: $$ex(n, S(K_s)) < \frac{1}{2} n^{2-\frac{1}{2}} + \frac{1}{2} p^2 n^{2+\frac{1}{2}}$$

for any integer $a \geq 2$. Select $a = \lceil \log_{1/p^2} n \rceil$. Then $$ex(n, S(K_s)) < \frac{1}{2} pm^2 + \frac{1}{2} p^2 n < mn^2$$

So if $G$ is a graph with at least $pn^2$ edges then $S(K_s) \subset G$ as required.
One cannot expect an $S(K_m)$ in a graph if $\binom{m}{2} + m > n$, and so the bound of order $n^{\frac{1}{2}}$ on $m$ in the above theorem is almost tight. With an additional idea, Alon, Krivelevich and Sudakov [1] showed that one can find $S(K_{\lceil p\sqrt{n} \rceil})$ in the above theorem, which is tight in the dependence on $p$.

### 2.3 Degenerate subgraphs

A graph $F$ is $r$-degenerate if it contains no subgraph of minimum degree at least $r + 1$. Erdős [4] conjectured that if $F$ is an $r$-degenerate bipartite graph, then $\text{ex}(n, F) = O(n^{2 - \frac{1}{r}})$. We also saw in Theorem 3 that this is true if every vertex in some part of $F$ has degree at most $r$. To approach the general case, we need a two-sided version of dependent random choice.

**Theorem 5.** Let $r, s, t \geq 1$, and let $G$ be bipartite graph with average degree $d$ and parts $U$ and $V$ of size $n$. Suppose

$$n^{r-s+s^2}d^{-s^2}(t-1)^s < \frac{1}{4}.$$  

Then there exist $X \subset U$ and $Y \subset V$ of size at least $4^{-\frac{1}{2}}d^sn^{1-s}$ each, such that every $X$ and $Y$ are $(t, r)$-rich in $G[X \cup Y]$.

**Proof.** Uniformly select sequences $S_U \in U^s$ and $S_V \in V^s$ such that the two sequences form the parts of a complete bipartite graph in $G$. By convexity, the number of such pairs of sequences is

$$N := \sum_{S \in U^s} d(S)^s \geq \sum_{S \in U^s} n^s \left( \frac{d(S)}{n^s} \right)^s \geq n^{s-s^2} \left( \sum_{v \in V} d(v)^s \right)^s \geq n^{2s-s^2} d^{s^2}.$$  

Let $X = N(S_U)$. A sequence $A \in X^r$ is $t$-poor if $N(A \cup S_V) < t$ and $A$ is $t$-rich otherwise. The expected number of $t$-poor sequences in $X^r$ is at most

$$\frac{1}{N} n^r(n-r)^s(t-1)^s < \frac{1}{4}$$  

by the inequality in the theorem. Therefore the probability that $X^r$ contains a $t$-poor sequence is less than $\frac{1}{4}$. A similar statement holds for $Y = N(S_V)$. We conclude
that with probability more than $\frac{1}{2}$, $X$ and $Y$ are $(r, t)$-rich. Also by convexity

$$E(|X|) = \frac{1}{n^s} \sum_{S \subseteq U^s} d(S) = \frac{1}{n^s} \sum_{v \in V} d(v)^s \geq \frac{d^s}{n^{s-1}} := D.$$ 

Therefore if $m = 4^{-\frac{1}{2}} D,$

$$\mathbb{P}(|X| < m) \leq \frac{1}{N} \sum_{S \subseteq U^s \atop d(S) < m} d(S)^s \leq \frac{n^s D^s}{4N} = \frac{1}{4}.$$ 

The same holds for $Y$, and therefore with probability more than $\frac{1}{2}$, we have $|X| \geq m$ and $|Y| \geq m$. So with positive probability, $X$ and $Y$ are the required sets. 

**Theorem 6.** Let $r \geq 2$, and let $F$ be an $r$-degenerate bipartite graph whose largest part has size $t$. Then

$$\text{ex}(n, F) \leq (t - 1)^{\frac{1}{2r}} n^2 - \frac{1}{4r}.$$ 

**Proof.** Let $G$ be an $n$-vertex $F$-free graph of average degree $2d$. Let $H$ be a balanced bipartite subgraph of $G$ of average degree at least $d$. Let $m = \lceil \frac{n}{2} \rceil$ and $s = 2r$. If

$$n^{r-s+s^2} d^{-s^2} (t-1)^s < \frac{1}{4} \quad \text{and} \quad 4^{-\frac{1}{2}} d^s n^{1-s} > t-1$$

then the last theorem provide sets $X, Y$ of size $t$ each that are both $(r, t)$-rich in $H[X, Y]$. The proof of the embedding lemma shows $F \subset H[X, Y]$, a contradiction. It follows that the above inequalities fail, and a calculation with $s = 2r$ gives

$$e(H) \leq \frac{1}{2} (t-1)^{\frac{1}{2r}} n^2 - \frac{1}{4r}.$$ 

Then $e(G) \leq 2e(H)$ so this gives the bound on $\text{ex}(n, F)$. 

It may be that we only require most $r$-subsets of $X$ and of $Y$ to be $t$-rich in order to embed $F$ in the last result. This leads to the following conjecture:

**Conjecture 7.** For $t, r \geq 2$, there exist $\delta > 0$ and $T$ such that if $G$ is a bipartite graph with parts $X$ and $Y$ of size $T$, and at least $(1 - \delta)T^r$ sequences in $X^r$ and in $Y^r$ are $t$-rich, then $G$ contains every bipartite $r$-degenerate graph with parts of size at most $t$.

This conjecture, if true, would give $\text{ex}(n, F) = O(n^{2-\frac{1}{4r+1}})$ for every $r$-degenerate bipartite graph $F$, using the proof of Theorem 6.
2.4 An application to Ramsey Numbers

The *Ramsey Number* of a graph $G$, denoted $r(G)$, is the minimum $n$ such that whenever $K_n$ is edge-colored with two colors, a monochromatic copy of $G$ appears. In other words, whenever $E(K_n) = E(G_1) \cup E(G_2)$, at least one of $G_1$ and $G_2$ contains $G$. Chvátal, Rödl, Trotter and Szemerédi [3] were the first to show that for each $d \geq 1$, there exists a constant $c(d)$ such that if $G$ has maximum degree $d$ and $n$ vertices, then $r(G) \leq c(d)n$. Since their proof uses Szemerédi’s Regularity Lemma, the dependence of $c(d)$ on $d$ is of tower type. Using dependent random choice, one can starkly improve this for bipartite graphs, based purely on density reasons.

**Theorem 8.** Let $H$ be a bipartite graph of maximum degree $r$ with $n$ vertices. Then $r(H) < 4^r(n+r)$.

**Proof.** If $E(K_N) = E(G_1) \cup E(G_2)$, we may suppose $e(G_1) \geq \frac{1}{2} \binom{N}{2}$. If $H$ is not a subgraph of $G_1$, then by Theorem 3 with $a = r$,

$$\frac{1}{2} \binom{N}{2} \leq \frac{1}{2} (n-1)^2 N^{2-\frac{1}{2}} + \frac{1}{2} (n + r - 1)^2 N^{2-\frac{1}{2}} < (n + r)^2 N^{2-\frac{1}{2}}.$$ 

This gives $N < 4^r(n+r)$.

From Theorem 5, we can obtain an upper bound on $r(H)$ when $H$ is $r$-degenerate and bipartite:

**Theorem 9.** Let $H$ be an $r$-degenerate bipartite graph with $n$ vertices. Then $r(H) \leq 2^{2\sqrt{r} \log n} n$.

**Proof.** If $E(K_N) = E(G_1) \cup E(G_2)$, we may suppose $e(G_1) \geq \frac{1}{2} \binom{N}{2}$. If $H$ is not a subgraph of $G_1$, apply Theorem 5 with $s = \lceil \sqrt{r \log n} \rceil$. A calculation gives the result.

Burr and Erdős [2] conjectured that if $H$ is any $r$-degenerate graph with $n$ vertices, then $r(H) \leq c(r)n$ for some constant $c$ depending only on $r$, and this conjecture remains open.
References


