Question 1.
Let $G$ be a graph on $n$ vertices containing no even cycle. Prove that $e(G) \leq \lfloor \frac{3}{2}(n - 1) \rfloor$ and classify the extremal graphs.

Question 2.
Let $G$ be a graph on $n$ vertices with at least $2n - 2$ edges. Prove that $G$ contains a subdivision of $K_4$ - a graph obtained by replacing each edge $\{u, v\}$ of $K_4$ with an arbitrary $uv$-path such that the paths are pairwise internally disjoint.

Question 3.
(a) Prove that every graph of minimum degree at least $d$ contains a cycle of length at least $d + 1$, and classify the extremal graphs of minimum degree $d - 1$ with no cycle of length at least $d + 1$.
(b) Prove Proposition 1 from the Notes Part 3.
(c) Determine $\text{ex}(n, \{C_3, P_k\})$ and determine the extremal graphs.

Question 4.
The aim of this question is to prove the theorem that every $n$-vertex graph with at least $\frac{1}{2}k(n - 1)$ edges contains a cycle of length at least $k + 1$.

(a) Let $k$ be a positive integer. Prove that an $n$-vertex graph with at least $\frac{1}{2}k(n - 1)$ edges has a block $B$ with at least $\frac{1}{2}k(|V(B)| - 1)$ edges.
(b) Let $G$ be a minimal counterexample to the theorem. Prove that $G$ is a block.
(c) Dirac’s Theorem states that a block of minimum degree at least $k$ has a cycle of length at least $2k$ or a hamiltonian cycle. Prove the theorem from Dirac’s Theorem.

Question 5.
If $G$ is a bipartite graph with parts $X$ and $Y$ with average degree $d_X \geq 2$ in $X$ and $d_Y \geq 2$ in $Y$, then the number of non-backtracking walks of length at most $k$ in $G$ is at least

$$|X| \sum_{i=0}^{k} (d_Y - 1)^{[i/2]}(d_X - 1)^{[i/2]} + |Y| \sum_{i=0}^{k} (d_X - 1)^{[i/2]}(d_Y - 1)^{[i/2]}.$$ 

(a) Prove that for all $m, n, k \geq 2$, $z(m, n, \mathcal{G}_{2k}) \leq (mn)^{1/2+1/4k} + m + n$.

(b) Prove that for all $m, n \geq 2$, $z(m, n, \mathcal{G}_{4}) \leq mn^{1/2} + m + n$.

(c) Let $G$ be a graph of girth at least five and let $X \subset V(G)$. Prove that if $G$ has minimum degree $d \geq 3\sqrt{|X|} + 3$, then $|\Gamma(X)| > 2|X|$.

(d) Show that if $G$ is a graph of girth five and average degree $d \geq 2$, then $G$ has a cycle of length $\Omega(d^2)$.

(e) Deduce $\text{ex}(n, \{C_3, C_4, P_k\}) = \Theta(\sqrt{kn})$.

**Question 6.**

(a) Prove that an $n$ by $n$ bipartite graph of minimum degree at least $n/2 + 1$ is hamiltonian.

(b) Prove that an $n$-vertex graph $G$ of minimum degree at least $n/2$ contains cycles $C_3, C_4, \ldots, C_n$ unless $n$ is even and $G = K_{n/2, n/2}$.

**Question 7**

Let $d \geq 2$. Prove that every $d$-regular graph with at most $2d + 1$ vertices is hamiltonian.

**Question 8.**

Let $\Gamma$ be an abelian group and suppose that $S \subset \Gamma$ is a set such that $a_1 + a_2 + \ldots + a_k = b_1 + b_2 + \ldots + b_k$ with $a_i, b_i \in S$ implies $\{a_1, a_2, \ldots, a_k\} = \{b_1, b_2, \ldots, b_k\}$. Call this a $k$-Sidon set. Let $G(\Gamma, S)$ be the bipartite graph whose parts are $A = B = \Gamma$ and where $\alpha \in A$ is adjacent to $\beta \in B$ if $\beta = \alpha + a$ for some $a \in S$.

(a) If $S$ is a 2-Sidon set, show that $G(\Gamma, S)$ is quadrilateral-free.

(b) Prove that if $q$ is an odd prime power and $\Gamma = \mathbb{F}_q^2$, then $S = \{(x, x^2) : x \in \mathbb{F}_q\}$ is a 2-Sidon set.

(c) Prove that if $S$ is a 3-Sidon set, then for every $\alpha \in A$ and $\beta \in B$, there are at most two $\alpha\beta$-paths of length three in $G(\Gamma, S)$.

(d) Prove that if $\Gamma = \mathbb{F}_q^3$ where $q \neq 3^t$ is a prime power, then $S = \{(x, x^2, x^3) : x \in \mathbb{F}_q\}$ is a generalized Sidon set.

(e) Prove that $n^{1/3} \leq z(n, n, \mathcal{F}) \leq 2^{1/3}n^{4/3}$ where $\mathcal{F}$ is the family of all graphs consisting of the union of three distinct paths of length three joining two vertices.

(f) If $S$ is the 4-Sidon set $\{(x, x^2, x^3, x^4) : x \in \mathbb{F}_q\}$ in $\Gamma = \mathbb{F}_q^4$, what is the maximum number of distinct paths of length four between two vertices of $G(\Gamma, S)$?
Question 9.

(a) Prove that the number of cycles of length four in any $n \times n$ bipartite graph with average degree $d$ is at least
\[
\binom{n}{2} \left( \frac{d(d-1)}{n-1} \right).
\]

(b) Let $G$ be a $d$-regular bipartite graph with parts $X, Y$ of size $n$. Prove that there exist two sets $U \subset X$ and $V \subset Y$ of size $d$ such that
\[
e(U, V) \geq (d - 1)\left( \frac{d(d-1)}{n-1} - 1 \right).
\]

(c) Prove that if $G$ is a $d$-regular $n \times n$ bipartite graph containing neither $K_{3,3}$ nor $Q_3$, then $e(G) \lesssim n^{8/5}$.

Question 10.

Prove that $\text{ex}(n, \{C_4, C_5\}) \sim \frac{1}{\sqrt{8}} n^{3/2}$ and $\text{ex}(n, \{C_4, C_6\}) \sim \frac{1}{2} n^{4/3}$.

Question 11.

Let $G$ be a bipartite graph whose parts are $\binom{A}{r}$ and $\binom{B}{r}$ where $|A| = |B| = n$. Suppose $G$ does not contain a complete bipartite graph whose parts are $\binom{A_1}{r}$ and $\binom{B_1}{r}$ with $|A_1| = |B_1| = s$. Prove that $e(G) \leq \binom{n}{r}^2 \left( 1 - \left( \frac{s}{r} \right)^2 \right)$.

Question 12.

Let $\mathcal{F}$ be the family of all 3-graphs consisting of four edges on at most seven vertices. Show that if $\text{ex}_3(n, \mathcal{F}) = o(n^2)$, then every set of integers that does not contain a 4-term arithmetic progression has zero density.

Question 13.

Let $d$ be a positive integer, and let $G$ be a graph of average degree at least $d$. Show that if
\[
\binom{d}{r} \geq T\left( \frac{t-1}{n} \right) \binom{n}{r}
\]
then there exists a set $Z \subset V(G)$ of at least $d$ vertices such that the number of $t$-poor sets in $\binom{Z}{r}$ is at most $\frac{1}{T} \binom{|Z|}{r}$.
Question 14.

(a) Prove that if $G$ is a 3-partite graph with parts $U, V, W$ of size $n$ such that $(U, V), (V, W), (U, W)$ are all $\varepsilon$-regular pairs of density more than $3\varepsilon$, then there exists an edge $\{u, v\}$ with $u \in U$ and $v \in V$ such that $u$ and $v$ have at least $\varepsilon^2 n$ common neighbors in $W$.

(b) Using the regularity lemma, prove that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $G$ is an $n$-vertex $K_4$-free graph with $(1 + \varepsilon)n^2/8$ edges, then $G$ contains an independent set of at least $\delta n$ vertices.

Question 15.

Using the regularity lemma, prove the Erdős-Stone Theorem: for any $\gamma > 0$, every graph with $(1 - 1/r + \gamma)\binom{n}{2}$ edges contains a complete $r$-partite graph with parts of size $\omega(n)$.

Question 16.

If $F$ is a graph, then the expansion of $F$ is the triple system whose edge set is $\{e \cup \{v_e\} : e \in E(F)\}$ where $v_e : e \in E(F)$ are distinct vertices disjoint from $V(F)$. Prove that if $H$ is a linear $n$-vertex 3-graph with $\delta n^2$ edges and $F$ is any 3-colorable graph, then if $n$ is large enough, $F \subseteq H$.

Question 17.

A loose triangle $C_3$ is a hypergraph of edges $\{a, b, c\}, \{c, d, e\}, \{e, f, a\}$ where $a, b, c, d, e, f$ are all distinct vertices. Prove that every $C_3$-free $n$-vertex 3-graph $H$ has at most $2e(\partial H)$ edges, and $\binom{n-1}{2} \leq \text{ex}_3(n, C_3) \leq n(n - 1)$.

Question 18*

Let $H$ be a linear 3-graph on $n$ vertices. Prove that if $H$ does not contain a loose 4-cycle i.e. the hypergraph $C_4$ consisting of edges $\{v_1, v_2, v_3\}, \{v_3, v_4, v_5\}, \{v_5, v_6, v_7\}, \{v_7, v_8, v_1\}$ with distinct vertices $v_1, v_2, \ldots, v_8$, then $e(H) = O(n^{3/2})$. Show that $\text{ex}_3(n, C_4) = \Theta(n^2)$. 


Hints.

2. First pass to a subgraph of minimum degree at least three and then first find a theta graph.
3c. Extremal graphs are $K_{\l,n-l}$ where $l = \floor{k/2}$. Make use of Proposition 1 in the Notes Part 3.
5c. Use $\text{ex}(n, C_4) \leq n^{3/2}$ and $z(m, n, \mathcal{C}) \leq mn^{1/2} + m + n$.
5d. Use Pósa’s Theorem (Theorem 5 in the Notes Part 3).
8d. Use breadth first search trees or non-backtracking walks to show $\text{ex}(n, \mathcal{F}) = O(n^{4/3})$.
9b. Consider an edge $\{u, v\}$ that is in a maximum number of quadrilaterals and set $U = N(u)$ and $V = N(v)$.
12. Define from a set $A$ with no 4-term progression a 3-graph $H_A$ as in the Ruzsa-Szemerédi Theorem and Behrend Construction.
13. Pick the neighborhood $Z$ of a uniformly randomly chosen vertex $w$, and consider $W - TX$ where $W = \binom{d(w)}{r}$ and $X$ is the number of $t$-poor subsets of $N(w)$.
14b. Take an $\varepsilon$-regular partition $V_0 \sqcup V_1 \sqcup V_2 \cdots \sqcup V_k$. If there is a triangle in the cluster graph consisting of $\varepsilon$-regular pairs of density more than $3\varepsilon$, then apply 14a. Otherwise, count the edges.
15. Use Turán’s Theorem in the cluster graph and the Embedding Lemma, and then carefully count edges.
16. Follow the proof of the Ruzsa-Szemerédi Theorem.
17. Show that $H$ has no 3-full subgraph.
18. Pass to a balanced 3-partite subgraph of $H$ with at least $\frac{2}{5}e(H)$ edges. If the parts are $X, Y, Z$ then consider the pairs $\{y, z\}$ in $\partial H$ such that $y \in Y$ and $z \in Z$ to be colored by the unique vertex $x \in X$ such that $\{x, y, z\} \in E(H)$. Show that a properly edge-colored $n$-vertex graph with no rainbow 4-cycle (a 4-cycle whose edges all have different colors) has $O(n^{3/2})$ edges.