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Notation: Let $f, g : \mathbb{N} \to \mathbb{R}$. We write $f \lesssim g$ to denote that $\limsup f(n)/g(n) \leq 1$ and $f \ll g$ to denote $\lim f(n)/g(n) = 0$.

6 Saddle Point Methods

In this section we show how to determine the asymptotics of coefficients of formal power series in $\mathbb{C}[[X]]$ with non-negative coefficients – these are the ones most frequently arising in combinatorics. The method is called the *saddle point method*, and it applies to contour integrals of functions $F(z) = A(z)/z^{n+1}$ which are analytic on some annulus $r_0 < |z| < r_1$ and such that for any $r \in (r_0, r_1)$, the unique maximum value of $|F(z)|$ on $|z| = r$ is $F(r)$. The most elementary *saddle point bound* is as follows:

Lemma 1 (Saddle Point Bound) *Let $n \in \mathbb{N}$ and let $A \in \mathbb{C}[[X]]$ be analytic at the origin with radius of convergence ρ and non-negative coefficients. Then for any $r \in (0, \rho)$,*

$$[X^n]A(X) \leq \frac{A(r)}{r^n}.$$

Proof ▷ Let γ be the circle $|z| = r$ where $s \in (0, \rho)$. By Cauchy's Integral Formula,

$$[X^n]A(X) = \frac{1}{2\pi i} \oint_\gamma \frac{A(z)}{z^{n+1}} dz.$$

Now the non-negativity of the coefficients implies the integral is at most

$$\frac{1}{2\pi} \oint_\gamma \sum_{k=0}^{\infty} a_n |z|^{k-n-1} dz \leq \frac{A(r)}{r^n}$$

and this completes the proof. ∎

A natural step is to choose the value of $r \in (0, \rho)$ that minimizes $A(r)/r^n$. There may be many such values of r in general, depending on what A looks like, and also one has to be careful on the boundary of the interval $(0, \rho)$. However the following lemma gives sufficient conditions for r to be unique. We refer to this lemma too as a saddle point bound.

Lemma 2 *Let $n \in \mathbb{N}$ and let $A \in \mathbb{C}[[X]]$ be analytic at the origin with radius of convergence ρ and non-negative coefficients. Suppose $A(x)/x^n \to \infty$ as $x \downarrow 0$ and $x \uparrow \rho$ and $(A'/A)' > 0$ for all $x \in (0, \rho)$. Then provided n is large enough, there exists a unique $r \in (0, \rho)$ such that $rA'(r) = nA(r)$ and*

$$[X^n]A(X) \leq \frac{A(r)}{r^n}.$$

Proof ▷ The condition $(A'/A)' > 0$ for $x \in (0, \rho)$ guarantees that $A(x)/x^n$ is unimodal and achieves a minimum at the unique value of r such that $rA'(r) = nA(r)$, if it exists at all in $(0, \rho)$. Now the existence of r is given by the fact that $A(x)/x^n \to \infty$ when $x \downarrow 0$ and $x \uparrow \rho$. The preceding lemma gives the result. ∎

The rôle of this lemma is to predict the best r to choose when applying saddle point bounds, and this value is called the *saddle point*. The equation

$$r\frac{A'(r)}{A(r)} = n$$

is called the *saddle point equation*. The bound in the first lemma holds for $n \in \mathbb{N}$, whereas the bound in the second lemma says that if n is large enough, say larger than the smallest m such that $\partial^m A(0) \neq 0$, then the bound applies. Already these simple results gives fairly good bounds on coefficients. The quality of the saddle point bound depends just how strongly the contribution to the integral drops off away from the saddle point; accordingly if $A(X) = 1/(1-X^2)^{1/2}$ or $A(X) = 1/(1-X)^{4/3}$ the quality of the approximation will not be as good as for say $A(X) = \exp(X/(1-X))$.

Central binomial coefficients and Stirling's Formula

Let $n \in \mathbb{N}$. The binomial coefficient $\binom{2n}{n}$ is $[X^n](1+X)^{2n}$. Applying the saddle point bound, and noting the saddle point equation

$$2nz(1+z)^{2n-1} = n(1+z)^{2n}$$

has unique solution $r = 1$, we get

$$\binom{2n}{n} \leq \frac{A(r)}{r^n} = 4^n.$$

Similarly for approximating $n!$, consider $1/n!$, which is $[X^n]\exp(X)$ and note the saddle point equation

$$re^r = ne^r$$

which has unique solution $r = n$, from which we obtain

$$n! \geq (n/e)^n.$$

Both of these estimates are out by a linear factor in $\sqrt{n}$, as we have seen before that

$$\binom{2n}{n} \sim \frac{4^n}{\sqrt{\pi n}}$$

and we also verified

$$n! \sim (n/e)^n\sqrt{2\pi n}.$$

Let $p(n)$ denote the number of partitions of n. Recall the generating function

$$\Phi(X) = \prod_{k=1}^{\infty}(1 - X^k)^{-1}.$$

This generating function has a strong singularity at $z = 1$, however note that it has other singularities at each complex root of unity. It is convenient to write

$$\Phi(X) = \exp\Big(\sum_{k=1}^{\infty} \frac{X^k}{k(1 - X^k)}\Big).$$

Factoring out $1 - X$ from each term in the denominator, we get

$$\Phi(X) = \exp\Big(\frac{1}{1 - X}\sum_{k=1}^{\infty} \frac{X^k}{kP_k(X)}\Big)$$

where $P_k(X) = 1 + X + \cdots + X^{k-1}$. The infinite series is absolutely convergent on the interior of the unit disc. So as real $z \uparrow 1$,

$$\log \Phi(z) \sim \frac{1}{1 - z} \cdot \sum_{k=1}^{\infty} \frac{1}{k^2} \sim \frac{\pi^2}{6(1 - z)} = \log f(z).$$

The saddle point bound for $f(z)$ occurs by picking

$$r = 1 - \Big(\frac{\pi}{6n}\Big)^{1/2}.$$

In this case, we quickly obtain

$$\log p(n) \lesssim \pi\sqrt{\tfrac{2n}{3}}.$$

This is only the saddle point bound. To use the saddle point method, we need to estimate the circle integral more carefully and in fact a full singularity analysis is required to obtain a series expansion for $p(n)$,

6.1 The Saddle Point Method

The idea of the saddle point method is to find coefficients of a formal power series $A(X) \in \mathbb{C}[[X]]$ by taking circle through the saddle point, applying Cauchy's Integral Formula, and then carefully estimating the integral over different parts of the circle. In generality, the main contribution to the circle integral comes from the part of the circle near the real axis, and the rest is negligible. This is not always the case, and sometimes one has to be much more careful when the integral has more than one peak – such as in the Hardy-Littlewood Circle Method. Here is how the general saddle point method works, and it works for general contour integrals. The set of conditions we have chosen is for simplicity, one can imagine many more general conditions under which the method still works (such as so-called Haiman admissibility - see later). For this section, it is convenient to redefine the saddle point as the solution to the saddle point equation

$$rA'(r) = (n+1)A(r)$$

which comes from the function $A(r)/r^{n+1}$.

Theorem 3 (Saddle Point Asymptotics I) *Let $A \in \mathbb{C}[[X]]$ and suppose $A(z)$ is analytic at the origin with saddle point r and let $f(z) = \log A(z)/z^{n+1}$. Let the circle γ of radius r be partitioned into $\gamma_1 = \{z : |\arg(z)| < \delta\}$ and $\gamma_2 = \{z : |\arg(z)| \geq \delta\}$ where:*

(1) $\displaystyle \frac{1}{rf''(r)^{1/2}} \ll \delta \ll \frac{1}{r^{2/3}f''(r)^{1/3}}.$

(2) *Uniformly on γ_1, $|f(z) - f(r) + \frac{1}{2}f''(r)r^2\arg(z)^2| = O(\delta^3 r^2 f''(r))$.*

(3) $\displaystyle \frac{1}{2\pi i}\int_{\gamma_2} e^{f(z)}dz \ll \frac{1}{2\pi i}\oint_{\gamma} e^{f(z)}dz.$

Then as $n \to \infty$,

$$[X^n]A(X) \sim \frac{e^{f(r)}}{\sqrt{2\pi f''(r)}} = \frac{A(r)}{r^{n+1}\sqrt{2\pi(\frac{n+1}{r^2} - \frac{A'(r)^2}{A(r)^2} + \frac{A''(r)}{A(r)})}}.$$

Proof ▷ By Cauchy's Integral Formula,

$$[X^n]A(X) = \frac{1}{2\pi i}\oint_{\gamma} e^{f(z)}dz = \frac{1}{2\pi i}\int_{\gamma_1} e^{f(z)}dz + \frac{1}{2\pi i}\int_{\gamma_2} e^{f(z)}dz.$$

We may ignore the second integral by (3). The first integral is, by Taylor's Theorem,

$$I = \frac{1}{2\pi i}\int_{\gamma_1} \exp(f(r) + f'(r)(z-r) + \tfrac{1}{2}f''(r)(z-r)^2 + \dots)dz.$$

Since r is the saddle point, $f'(r) = 0$. The change of variable $z \mapsto re^{i\theta}$ and (1) and (2) give

$$I \sim \frac{re^{f(r)}}{2\pi}\int_{-\delta}^{\delta} \exp(-\tfrac{1}{2}r^2 f''(r)\theta^2)d\theta.$$

Here we used the upper bound in (1) to deduce $\exp(\delta^3 r^2 f''(r)) \sim 1$ as $n \to \infty$. Make the change of variable $\theta \mapsto \theta/r\sqrt{f''(r)}$ and let $d = \delta r\sqrt{f''(r)}$. Then we get

$$I \sim \frac{e^{f(r)}}{2\pi\sqrt{f''(r)}}\int_{-d}^{d} e^{-\theta^2/2}d\theta.$$

By the lower bound in (1), $d \to \infty$ as $n \to \infty$. Therefore

$$I \sim \frac{e^{f(r)}}{2\pi\sqrt{f''(r)}} \int_{-\infty}^{\infty} e^{-\theta^2/2} d\theta = \frac{e^{f(r)}}{\sqrt{2\pi f''(r)}}.$$

This completes the proof. ∎

A simple way that (3) can be satisfied is if the function $F(z)$ drops extremely sharply as $\arg(z)$ moves away from zero, for instance when considering $F(z) = \exp(\exp(z) - 1)/z^{n+1}$ which comes from the Bell Numbers. However, if the nearest singularity of $A(z)$ to the origin is too weak, then the saddle point method can fail due to (3). For instance, if $A(X) = 1/(1 - X)$ then the saddle point method would predict the erroneous asymptotic formula $[X^n]A(X) \sim e/\sqrt{2\pi}$. We give a simple sufficient condition for (3) to hold:

Theorem 4 (Saddle Point Asymptotics II) *Let $A \in \mathbb{C}[[X]]$ and suppose $A(z)$ is analytic at the origin with saddle point r and let $f(z) = \log A(z)/z^{n+1}$. Let the circle γ of radius r be partitioned into $\gamma_1 = \{z : |\arg(z)| < \delta\}$ and $\gamma_2 = \{z : |\arg(z)| \geq \delta\}$ where $1 \gg \delta \gg \sqrt{2 \log r\sqrt{f''(r)}}/r\sqrt{f''(r)}$ as $n \to \infty$, and let $z_0 = re^{i\delta}$. Suppose that*

(1) *Uniformly on γ_1, $|f(z) - f(r) + \frac{1}{2}f''(r)r^2 \arg(z)^2| = o(r^2 f''(r) \arg(z)^2)$.*
(2) *Uniformly on γ_2, $|A(z)| \leq |A(z_0)|$.*

Then the saddle point asymptotic holds.

Proof ▷ On γ_2 we have

$$\left|\frac{1}{2\pi i} \int_{\gamma_2} e^{f(z)} dz\right| \leq r \frac{|A(z_0)|}{|z_0^{n+1}|}.$$

By (1), the preceding quantity is asymptotic to

$$r \exp(f(r) - \tfrac{1}{2}f''(r)r^2 \arg(z_0)^2) \leq r \exp(f(r) - \tfrac{1}{2}f''(r)\delta^2 r^2).$$

The saddle point asymptotic formula is

$$\frac{\exp(f(r))}{\sqrt{2\pi f''(r)}}.$$

Put $g(r) = r\sqrt{f''(r)}$. Then comparing this with the preceding bound, the integral over γ_2 is negligible provided

$$g(r) \exp(-\tfrac{1}{2}\delta^2 g(r)^2) \ll 1.$$

The choice of $\delta \gg \sqrt{2 \log g(r)}/g(r)$ ensures that this holds. ∎

Stirling's Formula

We recall

$$\frac{1}{n!} = \frac{e^n}{2\pi n^n} \int_{-\pi}^{\pi} \exp(n(e^{i\theta} - i\theta - 1))d\theta.$$

Here the saddle point is $r = n + 1$ and let's choose δ from the preceding theorem to be $n^{-2/5} \gg \sqrt{2 \log g(r)}/g(r)$. Then for $\pi \geq |\theta| \geq \delta$, and $A(z) = \exp(z)/z^{n+1}$,

$$|A(z)| = |\exp((n+1)e^{i\theta})| = \exp((n+1)\cos\theta) \leq \exp((n+1)\cos\delta) = |A(z_0)|$$

and so (2) in the preceding theorem is satisfied. Note that for $|\theta| < \delta$ and $z = ne^{i\theta}$,

$$f(z) - f(n+1) = -\tfrac{\arg(z)^2}{2}(n+1) + O(n\arg(z)^3) = -\tfrac{\arg(z)^2}{2}(n+1) + o(n\arg(z)^2)$$

since on γ_1 we have $\arg(z) \leq \delta$. It follows that the condition (1), namely

$$|f(z) - f(r) + \tfrac{1}{2}f''(r)r^2\arg(z)^2| = O(\delta^3 r^2 f''(r)) = o(r^2 f''(r)\arg(z)^2)$$

is met uniformly on γ_1. The saddle point asymptotic is

$$\frac{1}{n!} \sim \frac{e^n}{\sqrt{2\pi n}n^n}$$

as required for Stirling's Formula. We remark that here the error term coming from the integral over γ_2 is very small indeed – roughly $\exp(-\delta^2 n) = \exp(-n^{1/6})$. So the lower order terms in Stirling's Formula will come from the approximations of the integral over γ_1.

Involutions

An involution is a permutation σ with σ^2 equal to the identity. The exponential generating function for involutions is $\exp(X + X^2/2)$. For this example $f(z) = z + z^2/2 - (n+1)\log z$. The third order term in the expansion of $f(z)$ looks like $\theta^3 r^3 f'''(r)$. We have $f''(r) \sim 2$ and $f'''(r) = O(n^{-1/2})$ and so we can take $\delta = n^{-2/5}$ to get (1) in the theorem. We leave (2) as an exercise, so the saddle point asymptotic formula holds and as an exercise one verifies that the number of involutions is asymptotic to

$$\frac{e^{\sqrt{n}-1/4}}{2\sqrt{\pi n}}\left(\frac{e}{n}\right)^{n/2}.$$