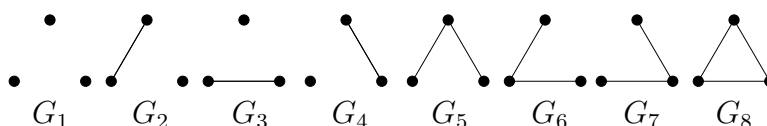


ASSIGNMENT 1 | INTRODUCTION TO GRAPH THEORY | CO342

This assignment will be marked out of a total of thirty points, and is due on Thursday 18th May at 10am in class. Throughout the assignment, the graphs are all finite undirected graphs, without multiple edges or loops, unless specifically stated.

Hand-in Questions.

- There are eight graphs on three vertices, and they are shown below.



- Determine the number of graphs on n vertices. [3]
 - Determine the number of graphs on n vertices with exactly m edges. [3]
- Prove that every graph with at least two vertices contains a pair of distinct vertices of the same degree. Determine whether this also holds for multigraphs, by providing a proof or a counterexample. [4]
 - Let $G_n = (V_n, E_n)$ denote the graph with vertex set $V_n = \{0, 1, 2, \dots, n-1\}$ and edge set $E_n = \{\{i, j\} : i + j = 1 \pmod n\}$, where $n \geq 3$.
 - Draw the graphs G_3, G_4 and G_5 . [3]
 - How many components does G_n have for $n \geq 3$? [2]
 - Prove that for any graph G , $\kappa(G) \leq \lambda(G) \leq \delta(G)$. [3]
 - Find an example of a graph G where $\kappa(G) = 2$, $\lambda(G) = 3$ and $\delta(G) = 4$. [3]
 - Let G be a graph of maximum degree at most three. Prove that $\lambda(G) = \kappa(G)$. [5]
Is this true for graphs of maximum degree at most four? Provide a proof or a counterexample.
 - Let G be a 2-connected graph on at least four vertices. Prove that for any edge $e \in E(G)$, G/e is 2-connected or $G - e$ is 2-connected. [4]

Bonus Question.

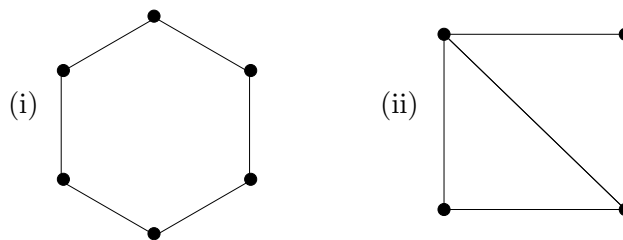
- Let G be a 2-connected graph with m edges and n vertices. Prove that G contains at least $m - n + 2$ distinct paths connecting u and v for any pair of distinct vertices $u, v \in V(G)$, and at least $\binom{m-n+2}{2}$ distinct cycles. [6]

[Hint for Question 7: use the ear-decomposition theorem for 2-connected graphs, and prove both statements by induction on m .]

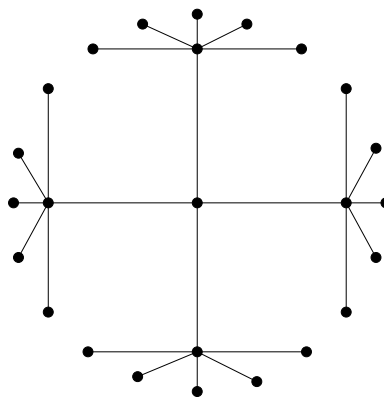
This assignment will be marked out of a total of thirty points. Throughout the assignment, the graphs are all finite undirected graphs, without loops or multiple edges. The assignment is to be handed in on Thursday June 1st in class.

Hand-in Questions.

1. (a) Let G be a k -connected graph, and let $A \subset V(G)$ have size k . Let H be obtained from G by adding a new vertex adjacent to all vertices in A . Prove that H is k -connected. [3]
- (b) Let u, v be non-adjacent vertices in a graph G , and W a minimum set of vertices separating u from v . Let H be a component of $G - W$ containing u , and let w be the vertex of $G/V(H)$ to which $V(H)$ is contracted. Prove, without Menger's Theorem, that the minimum size of a set of vertices in $G/V(H)$ separating w from v is $|W|$. [3]
2. (a) The **line graph** of a graph $G = (V, E)$ is the graph $L(G)$ with vertex set E and edge set $F = \{\{e, f\} \subset E : e \cap f \neq \emptyset\}$. Draw the line graphs of each of the two graphs illustrated below: [4]



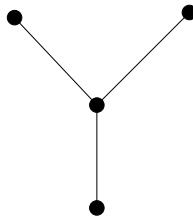
2. (b) Prove the edge-form of Menger's Theorem stated in the notes by applying the vertex form of Menger's Theorem to line graphs. [3]
- (c) Let G be any graph on twenty-five vertices consisting of four edge-disjoint copies of the tree shown below. Determine the value of $\lambda(G)$, and justify your answer. [2]



3. Prove that every 3-connected graph contains a subdivision of K_4 . [4]

Hand-in Questions continued

4. Let G be an r -regular bipartite graph, and let L be a set of $r - 1$ edges of G . Prove that $G - L$ has a perfect matching using Hall's Theorem. [4]
5. (a) Let G be a graph. Prove that $L(G)$, the line graph of G , cannot contain the following graph as an induced subgraph. [2]



5. (b) Use Tutte's 1-Factor Theorem to prove that if G is a connected graph with an even number of edges and no bridges, then $L(G)$ has a perfect matching. [3]
6. (a) A graph G is **t -expanding** if each independent set $I \subset V(G)$ has at least $t|I|$ neighbours. Prove that every t -expanding graph on n vertices has a matching which covers at least $n - n/(t + 1)$ vertices. [3]
- (b) For each even integer $t \geq 2$, find a t -expanding graph on $2t + 2$ vertices containing no matching covering more than $n - n/(t + 1)$ vertices. [1]

Bonus Question.

7. A square matrix A is **doubly stochastic** if each of its entries is non-negative and every row and every column of A sums to one. A **permutation matrix** is a matrix with exactly one non-zero entry in each row and column. Prove that if A is a doubly stochastic, then [4]

$$A = \lambda_1 P_1 + \lambda_2 P_2 + \dots + \lambda_k P_k$$

where each P_i is a permutation matrix, and $\lambda_i \geq 0$ are reals satisfying $\sum_{i=1}^r \lambda_i = 1$.

[Hint : Use Hall's Theorem to find λ_1, P_1 such that $A - \lambda_1 P_1$ has fewer non-zero entries than A , and proceed by induction on the number of non-zero entries of A .]

This assignment will be marked out of a total of thirty points. The assignment is to be handed in on Thursday 15th June in class.

Hand-in Questions.

1. Let G be a graph of minimum degree at least $2d - 1$, where $d \geq 1$. Prove that G contains a matching of size d . Also, for each $d \geq 1$, find a graph of minimum degree at least $2d - 1$ with no matching of size $d + 1$. [3]
2. Let G be a k -edge-connected k -regular graph with an even number of vertices. Using Tutte's 1-Factor Theorem, prove that G has a perfect matching. [3]
3. Let $G(A, B)$ be a bipartite graph such that $|B| = k|A|$, and define [3]

$$f(v) = \begin{cases} k & \text{if } v \in A \\ 1 & \text{if } v \in B. \end{cases}$$

Prove that G has an f -factor if and only if $|\Gamma(X)| \geq k|X|$ for each set $X \subset A$.

4. (a) Let $\delta : V \rightarrow \mathbb{N}$ be any function. An orientation of graph $G = (V, E)$ is called a δ -**orientation** if each $v \in V$ has outdegree $\delta(v)$. Draw a δ -orientation of the graph shown in Figure 1, when $\delta(v) = \delta(w) = \delta(y) = \delta(z) = 1$ and $\delta(x) = 2$. [3]

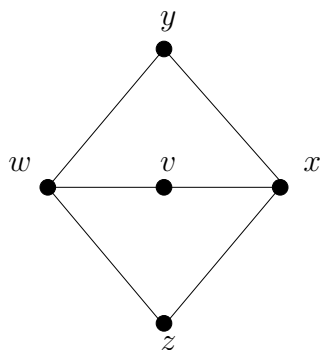


Figure 1

- (b) Let G^* be the graph obtained from a graph G by subdividing each edge of G exactly once, and define $f : V(G^*) \rightarrow \mathbb{N}$ by $f(v) = \delta(v)$ for $v \in V(G)$ and $f(v) = 1$ otherwise. Show that G has a δ -orientation if and only if G^* has an f -factor. [3]

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Hand-in Questions continued

4. (c) Using the bipartite f -factor theorem, prove that a graph $G = (V, E)$ has a δ -orientation if and only if for every set $X \subseteq V$, [3]

$$(1) \quad \sum_{v \in V} \delta(v) = |E|$$

$$(2) \quad \sum_{x \in X} \delta(x) \leq e(X, V)$$

where $e(X, V)$ is the set of edges of G incident with at least one vertex of X .

5. (a) Let G be a bipartite graph of maximum degree k . Construct a k -regular bipartite multigraph H containing G , and with twice as many vertices as G . [3]

(b) Let G be a bipartite graph of maximum degree k with no isolated vertices. Prove that G is a union of k edge-disjoint matchings. [3]

(c) Let $k \geq 1$ be an integer, and let $G(A, B)$ be a bipartite graph such that $f(v) = d(v)/k$ is an integer, for all $v \in V(G)$. Show that $G(A, B)$ has an f -factor. [3]

6. Let G be a connected graph in which every vertex has even degree and such that the number of edges is even. Let $f(v) = d(v)/2$ for every $v \in V(G)$. Using Euler's Theorem, prove that G has an f -factor. [3]

Bonus Question.

7. Using the max flow min cut theorem, prove that the conditions in the bipartite f -factor theorem are sufficient for an f -factor, by adding a source and sink and appropriate arc capacities. [4]

This assignment will be marked out of a total of thirty points. Throughout the assignment, the graphs are all finite undirected graphs, without multiple edges or loops.

1. Let $G = G_0$ be a graph. Prove that the following algorithm determines the degeneracy of G : while $V(G_i)$ is not empty, select a vertex $v_i \in V(G_i)$ of smallest degree, say d_i , and let $G_{i+1} = G_i - \{v_i\}$. Then output $d(G) = \max\{d_i : i = 0, 1, 2, \dots\}$. [3]
2. Prove that $\chi'(K_{2n+1}) = n$ for $n \geq 1$. [3]
3. Prove that $\chi'(K_{2n}) = 2n - 1$ for all $n \geq 1$. [5]
4. Let G_1 and G_2 be graphs and suppose that $G_1 \cap G_2$ is a complete graph. Show that $\chi(G_1 \cup G_2) \leq \chi(G_1) + \chi(G_2) - \chi(G_1 \cap G_2)$. [3]
5. (a) Show $d(G) \leq 3$ when G is planar and has no triangles, and deduce $\chi(G) \leq 4$. [3]
 (b) Find a planar graph G with $d(G) = 4$ and $\chi(G) = 4$. [3]
6. A decomposition of a graph G into bipartite graphs is a collection of edge-disjoint bipartite graphs B_1, B_2, \dots, B_k such that $G = B_1 \cup B_2 \cup \dots \cup B_k$. Prove the following:
 - (a) if G has a decomposition into k bipartite graphs, then G is 2^k -colourable. [3]
 - (b) K_{2^k} has a decomposition into k bipartite graphs. [3]
 - (c) K_{2^k+1} does not have a decomposition into k bipartite graphs. [1]
7. Let G be a plane graph whose infinite face is bounded by a cycle of length $n = |V(G)|$. Prove that G is 3-colourable. [3]

Bonus Question.

8. Let X be a set of $2^k + 1$ points in the Euclidean plane. Prove that there exist three points in X making a triangle whose largest angle has size at least $\pi(1 - 1/k)$.

[Hint: for $0 \leq i < k$, define a graph B_i whose edge set is

$$E_i = \{\{a, b\} \subset X : i\pi/k \leq \theta(a, b) < (i + 1)\pi/k\}$$

where $\theta(a, b)$ denotes the angle between the line through a and b and the x -axis, taken counterclockwise from the x -axis to the line. Show using Question 6(c) that some B_i has an odd cycle, and that this gives the required three points in X .] [5]

This assignment is to be handed in on Thursday July 20th in class, and will be marked out of twenty points, though it carries the same weight as all the other assignments in the computation of the final grade.

Hand-in Questions.

1. A maximal plane graph is a plane graph $G = (V, E)$ with $n \geq 3$ vertices such that if we join any two non-adjacent vertices in G , we obtain a non-plane graph.
 - (a) Draw a maximal plane graph on six vertices. [3]
 - (b) Show that a maximal plane graph on n vertices has $3n - 6$ edges. [4]
 - (c) Prove that every maximal plane graph on $n \geq 4$ vertices is 3-connected. [3]

2. (a) Prove that if G is a graph of girth g which can be embedded in a surface of Euler characteristic χ without crossings, then $|E(G)| \leq g(|V(G)| - \chi)/(g - 2)$. [3]
 - (b) Use (a) to prove that the graph in Figure 2 is not toroidal. [2]

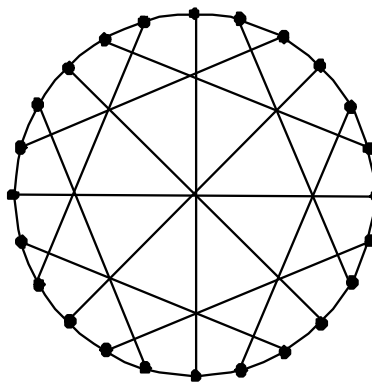


Figure 2

3. Prove that every graph with $n \geq 4$ vertices and at least $\lfloor n^2/4 \rfloor + 1$ edges contains two triangles sharing one edge. [7]
 [Hint: induction on n ; delete a vertex of lowest degree.]