Let $T_r(n)$ denote a complete $r$-partite $n$-vertex graph whose parts have sizes as equal as possible. These are called Turán graphs. When $r = 2$, we obtain precisely the extremal triangle-free graphs $K_{\lceil n/2 \rceil, \lfloor n/2 \rfloor}$. The following lemma regarding the Turán graphs is left as an exercise:

Lemma 4.1. Let $n \geq r \geq 2$.

(a) $e(T_r(n)) \sim (1 - \frac{1}{r-1}) \binom{n}{2}$.
(b) If $G$ is an $r$-partite $n$-vertex graph, then $e(G) \leq e(T_r(n))$.
(c) If $G$ is any $n$-vertex graph with $e(G) = e(T_r(n))$, then $\delta(G) \leq \delta(T_r(n))$.

These observations are key to proving Turán’s Theorem, which shows $\text{ex}(n, K_{r+1}) = e(T_r(n))$.

Theorem 4.2. (Turán, 1941) $\text{ex}(n, K_{r+1}) = e(T_r(n))$ and the unique extremal $K_{r+1}$-free graphs are Turán graphs

Proof. Proceed by induction on $n$. We show that if $G$ is an $n$-vertex $K_{r+1}$-free graph with at least $e(T_r(n))$ edges, then $G = T_r(n)$. This is clear if $n \leq r$, for in this case $T_r(n)$ is a complete graph. If $n > r$, and $G$ is an $n$-vertex graph with at least $e(T_r(n))$ edges, delete edges until we get $H \subseteq G$ with $e(H) = e(T_r(n))$. By part (c) of the lemma, $\delta(H) \leq \delta(T_r(n))$. Since $e(T_r(n)) = e(T_r(n-1)) + \delta(T_r(n))$, if $v$ is a vertex of minimum degree in $H$ then $e(H - \{v\}) \geq e(T_r(n-1))$. By induction, $H - \{v\} = T_r(n-1)$, and now $v$ must be added to a smallest part of $H$, which gives $H = T_r(n)$. Since $T_r(n)$ is maximal $K_{r+1}$-free (i.e. no edge can be added without creating $K_{r+1}$), and $T_r(n) = H \subseteq G$, we must have $G = T_r(n)$. \qed

4.1 The Erdős-Stone Theorem

The following theorem of Erdős and Stone (1946) was observed by Simonovits to give the asymptotic value of $\text{ex}(n, F)$ whenever $F$ is not bipartite.

Theorem 4.3. (Erdős-Stone, 1946) Let $r \geq 0$. For all $\varepsilon > 0$ and $k \geq 1$, there exists $n_r(\varepsilon, k)$ such that if $G$ is an $n$-vertex graph with at least $(1 - 1/r + \varepsilon) \binom{n}{2}$ edges, then $G$ contains $T_{r+1}(k)$. 

Before we prove this, let’s determine the asymptotic behavior of $\text{ex}(n,F)$ for every graph $F$ of chromatic number $\chi(F) = r + 1$. The chromatic number of $F$ plays the key role: recall the chromatic number $\chi(F)$ is the smallest $r$ such that we may write $V(F) = V_1 \cup V_2 \cup \cdots \cup V_r$ and every $V_i$ contains no edges of $F$ (i.e. the $V_i$ are independent or stable sets). Usually, we refer to the vertices in $V_i$ as the vertices of color $i$, and the partition is referred to as a proper $r$-coloring of $F$. So the chromatic number is the smallest $r$ such that there is a proper $r$-coloring of $F$.

As is evident, $\chi(T_r(n)) = r$ for $n \geq r$. Note that $\chi(F) > r$, then $T_r(n)$ does not contain $F$. Therefore $\text{ex}(n,F) \geq c(T_r(n))$. Since $c(T_r(n)) \sim (1 - 1/r)\binom{n}{2}$ as $n \to \infty$, we have
\[
\text{ex}(n,F) \geq (1 - 1/r)\binom{n}{2} - o(n^2).
\]

On the other hand, if $k$ is sufficiently large, then $F \subset T_{r+1}(k)$. By the Erdős-Stone Theorem, for any $\epsilon > 0$, if $G$ is an $n$-vertex graph with at least $(1 - 1/r + \epsilon)\binom{n}{2}$ edges and $n \geq n_\epsilon(r,k)$, then $T_{r+1}(k) \subset G$. It follows that $\text{ex}(n,F) \leq (1 - 1/r)\binom{n}{2} + o(n^2)$ as $n \to \infty$, and together with the lower bound on $\text{ex}(n,F)$, we obtain the following theorem:

**Theorem 4.4. (Simonovits, 1966)** Let $F$ be a graph with $\chi(F) = r \geq 2$. Then as $n \to \infty$,

\[
\text{ex}(n,F) = \left(1 - \frac{1}{r-1}\right)\binom{n}{2} + o(n^2). \tag{4.1.1}
\]

In the case $r = 2$, i.e. $F$ is bipartite, (4.1.1) only gives $\text{ex}(n,F) = o(n^2)$, so the asymptotic value of $\text{ex}(n,F)$ when $F$ is bipartite is not determined by this theorem. This is the notoriously difficult bipartite Turán problem, which we shall study in detail soon.

The Turán density $\pi(F)$ for a graph $F$ is defined by $\pi(F) = \lim_{n \to \infty} \text{ex}(n,F)/\binom{n}{2}$. The above theorem determines the Turán density of every graph. As an exercise, the reader may determine the Turán density of each of the platonic solid graphs.

### 4.2 Proof of Erdős-Stone Theorem.

The proof of the Erdős-Stone Theorem is somewhat technical. We do not spend time to determine the best possible function dependence of $n_\epsilon(r,k)$ on $r, \epsilon$ and $k$. As it turns out, Bollobás sharpened the Erdős-Stone Theorem in the following manner, which gives the right order of magnitude of $k$ in terms of $n$:

**Theorem 4.5. (Bollobás, 1994)** For all $r \geq 1$, $\epsilon > 0$, there exists $c(r,\epsilon) > 0$ such that if $G$ is any $n$-vertex graph with at least $(1 - 1/r + \epsilon)\binom{n}{2}$ edges, then $G$ contains $T_r(k)$ where $k \geq c(r,\epsilon) \log n$.

This result is tight up to the value of the constant $c(r,\epsilon)$: the reader may verify the statement that for every $\delta > 0$ and $r \geq 2$, there exists $C = C(r,\delta) > 0$ such that with positive probability, the random graph $G_{n,1-\delta}$ has at least $(1 - \delta)\binom{n}{2}$ edges and does not contain $T_r(C \log n)$.

**Lemma 4.6.** Let $c, \epsilon > 0$ and let $G$ be an $m$-vertex graph with $(c + \epsilon)\binom{m}{2}$ edges. Then $G$ has a subgraph with $n \geq \sqrt{cm}/4$ vertices and minimum degree at least $(c + \epsilon/2)(n - 1)$.
Comparing the bounds on vertex of degree less than $(c + \epsilon/2)(m - i - 1)$, remove it and continue to $G_{i+1}$. Then for any $i$,

$$\binom{m-i}{2} \geq e(G_i) > (c + \epsilon)\binom{m}{2} - \sum_{j=1}^{m-1} (c + \epsilon/2)j > (\epsilon/2)\binom{m}{2}.$$  

It follows that $m - i \geq \sqrt{\epsilon m}/4$. In other words, the process must stop at some $G_i$ with $n = m - i \geq \sqrt{\epsilon m}/4$ vertices. Now we have $\delta(G_i) \geq (c + \epsilon/2)(n - 1)$, as required.  

**Proof of Erdős-Stone.** By induction on $r$. If $r = 0$, the result is easy since $T_1(k)$ is just a set of $k$ vertices and no edges, and so $n_0(\varepsilon, k) = k$. Suppose $G$ is an $m$-vertex graph with $e(G) \geq (1 - 1/r + \epsilon)\binom{m}{2}$. Via the lemma, pass to an $n$-vertex subgraph $H$ of $G$ such that $\delta(H) \geq (1 - 1/r + \epsilon/2)n$ and $n \geq \sqrt{\epsilon m}/4$. By induction, $H$ contains $T_r(K)$ if $n \geq n_{r-1}(\varepsilon/2, K)$. Let $T = V(T_r(K))$ and let $V_1, V_2, \ldots, V_r$ be the parts of $T_r(k)$, each of size at least $\ell = \lceil K/r \rceil$. Let $U$ be the set of vertices in $V(G) \setminus T$ which have at least $k/r$ neighbors in each of the sets $V_i$. If $W = V(G) \setminus T$, then

$$e(T, W) = e(T, U) + e(T, W \setminus U) \leq |U|K + (n - |U|) (K - \ell + k/r - 1).$$

On the other hand, since $\delta(G) \geq (1 - 1/r + \epsilon/2)n$,

$$e(T, W) \geq (1 - 1/r + \epsilon/2)nK - K^2.$$  

Comparing the bounds on $e(T, W)$, we get

$$|U| \geq \epsilon n/2 - kn/K - K.$$  

If $K = \lceil 4k/\epsilon \rceil$ and $n$ is large enough, this gives $|U| \geq \epsilon n/4$. To each vertex $u \in U$, associate an $r$-tuple $(U_1, U_2, \ldots, U_r)$ where $U_i \subseteq N(u) \cap V_i$ and $|U_i| = \lceil k/r \rceil$. There are at most $\binom{K}{r}/r$ such $r$-tuples, and so if $n$ is large enough to ensure

$$\frac{\epsilon n}{4} \geq \frac{k}{r} \binom{\lceil K/r \rceil}{r},$$

then there must be a set $U_0$ of at least $k/r$ vertices of $U$ each of which is associated with the same $r$-tuple $(U_1, U_2, \ldots, U_r)$. So $U_1 \cup U_2 \cup \cdots \cup U_r \subset N(u)$ for all $u \in U$, which means $U_0, U_1, \ldots, U_r$ are the parts of a Turán graph with at least $k$ vertices in $G$, as required.  

In the special case $r = 2$, Bollobás’ Theorem is less difficult to prove: the reader may prove that if $\delta > 0$ and $k \leq \delta n$ is a positive integer, then if $G$ is an $n$-vertex graph with $\delta n^2$ edges, $G$ contains a complete bipartite graph $K_{k,m}$ where $m = \lceil \delta^k n \rceil$.  

3
4.3 Kövari-Sós-Turán Theorem

The Erdős-Stone Theorem gives $\text{ex}(n, F) \sim (1 - 1/(\chi(F) - 1))(\binom{n}{2})$ for any graph $F$, so we have asymptotics when $\chi(F) \geq 3$ but not when $\chi(F) = 2$ i.e. $F$ is bipartite. We focus on this case now, and this lecture looks at $F = K_{s,t}$, the complete bipartite graph with parts of sizes $s$ and $t$. The extremal problem for $K_{s,t}$ is related to the Zarankiewicz problem (1951). This problem asks for the maximum number $z(m, n, s, t)$ of 1s in an $m \times n$ 0-1 matrix that does not contain an $s \times t$ minor filled with 1s. Observe that if $A$ is an $m \times n$ 0-1 matrix, then $A$ is the incidence matrix of an $m \times n$ bipartite graph: the rows of the matrix are labelled by the vertices in the part of size $m$, and the columns are labelled by the vertices in the part of size $n$. If $A$ has no $s \times t$ minor filled with 1s, then the graph has no $K_{s,t}$ with $s$ vertices in the part of size $m$ and $t$ vertices in the part of size $n$.

**Theorem 4.7. (Kövari-Sós-Turán, 1954)** For $m, n, s, t \geq 1$,

$$z(m, n, s, t) \leq (s - 1)^{1/t}n^{m-1/t} + (t-1)m. \quad (4.3.1)$$

**Proof.** We write the proof in the language of graphs. Let $A$ be an $m \times n$ matrix with $z = z(m, n, s, t)$ ones and no $s \times t$ minor filled with 1s, and let $G$ be the corresponding $m \times n$ bipartite graph, say with parts $R$ of size $m$ and $C$ of size $n$. They key observation is

$$\sum_{v \in R} \binom{d(v)}{t} \leq (s-1)\binom{|C|}{t}.$$

By convexity, this implies

$$|R|\left(\frac{|E(G)|}{|R|}\right) \leq (s-1)\binom{|C|}{t}.$$

Using $(x - t + 1)^t/t! \leq (\frac{x}{t})^t \leq x^t/t!$ for $x \geq t - 1$, we get

$$m(z/m - t + 1)^t \leq (s-1)n^t.$$

This gives (4.3.1). \qed

The proof of the bound above also gives an upper bound on the Turán numbers $\text{ex}(n, K_{s,t})$:

**Theorem 4.8.** For $s,t \geq 1$,

$$\text{ex}(n, K_{s,t}) \leq \frac{1}{2} (s-1)^{1/t}n^{2-1/t} + \frac{1}{2} (t-1)n. \quad (4.3.2)$$

**Proof.** If $G = (V, E)$ is a $K_{s,t}$-free $n$-vertex graph, then

$$\sum_{v \in V} \binom{d(v)}{t} \leq (s-1)\binom{n}{t}.$$

Since $\sum_{v \in V} d(v) = 2e(G)$, the convexity of binomial coefficients gives

$$n(2e(G)/n - t + 1)^t \leq (s-1)n^t.$$
which gives the theorem. □

In general, the order of magnitude of $\text{ex}(n, K_{s,t})$ and $z(m, n, s, t)$ are not known when $s, t \geq 4$, and specifically, $K_{4,4}$. In the special case $s = t = 2$, much more is known using projective geometry.

### 4.4 Projective spaces

For a vector space $V$, let $\begin{bmatrix} V \end{bmatrix}_k$ denote the set of all $k$-dimensional subspaces of $V$. Fixing a prime power $q \geq 2$ and a positive integer $n$, define the Gaussian binomial coefficients:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \prod_{i=0}^{k-1} \frac{q^{n-i} - 1}{q^{k-i} - 1}. \quad (4.4.1)$$

These are called Gaussian binomial coefficients. When $q$ is clear from the context, we write $\begin{bmatrix} n \\ k \end{bmatrix}$ instead of $\begin{bmatrix} n \\ k \end{bmatrix}_q$. The Gaussian binomial coefficients count $k$-dimensional subspaces of an $n$-dimensional vector space over $\mathbb{F}_q$.

**Lemma 4.9.** If $V$ is an $n$-dimensional vector space over a finite field $\mathbb{F}_q$, then

$$\left| \begin{bmatrix} V \\ k \end{bmatrix} \right| = \begin{bmatrix} n \\ k \end{bmatrix}. \quad (4.4.2)$$

The elements of $\begin{bmatrix} V \\ 1 \end{bmatrix}$, $\begin{bmatrix} V \\ 2 \end{bmatrix}$ and $\begin{bmatrix} V \\ 3 \end{bmatrix}$ are referred to as points, lines and planes respectively. For any $n \geq k \geq \ell$ and an $n$-dimensional vector space $V$ over $\mathbb{F}_q$, let $H_q[n, k, \ell]$ be the hypergraph whose vertex set is $\begin{bmatrix} V \end{bmatrix}_\ell$ and whose edge set is the set of $\begin{bmatrix} W \end{bmatrix}_\ell$ such that $W$ is a subspace of $V$ of dimension $\ell$. The statistics are described by the following lemma, following from Lemma 4.9.

**Lemma 4.10.** For $n \geq k \geq \ell$, the hypergraph $H_q[n, k, \ell]$ is a $\begin{bmatrix} k \\ \ell \end{bmatrix}_q$-uniform $\begin{bmatrix} n-\ell \\ k-\ell \end{bmatrix}_q$-regular hypergraph with $\begin{bmatrix} n \\ \ell \end{bmatrix}_q$ vertices.

The hypergraph $H_q[n + 1, 2, 1]$ is a combinatorial representation of the $n$-dimensional projective space of order $q$, denoted $\text{PG}(n, q)$. We refer to this as a projective plane when $n = 2$. We draw below a picture of the Fano plane $\text{PG}(3, 2)$:

![Fano plane](image1)

![Heawood graph](image2)

Figure 2: Fano plane and Heawood graph $G_2[3, 2, 1]$
We let $G_q[n,k,\ell]$ be the bigraph of $H_q[n,k,\ell]$, namely one part of the bipartite graph represents the points, one part represents the lines, and a point is joined to a line if the line contains that point.

**Theorem 1.** For any prime power $q$, $G_q[3,2,1]$ is a $(q+1)$-regular graph of girth six and $z(q^2 + q + 1, q^2 + q + 1, 2, 2) = e(G_q[3,2,1])$.

**Proof.** It only has to be shown that $G_q[3,2,1]$ has no cycles of length four. If $U,W$ are two-dimensional subspaces of the three-dimensional vector space $V$ over $\mathbb{F}_q$, then $\dim(U \cap W) = 1$. It follows that every pair of vertices of $[\frac{V}{2}]$ in $G_q[3,2,1]$ have exactly one common neighbor in $[\frac{V}{1}]$, and therefore $G_q[3,2,1]$ contains no quadrilaterals. Therefore $G_q[3,2,1]$ has girth six. A counting argument shows the graphs $G_q[3,2,1]$ are in fact extremal bipartite graphs with no 4-cycles. $\square$

We now show how to get extremal (non-bipartite) graphs without cycles of length four. Let $V$ be a three-dimensional vector space over $\mathbb{F}_q$. Let $ER_q$ be the graph with $V(ER_q) = [\frac{V}{1}]$ and the edge-set of $ER_q$ is the set of pairs $(U,W)$ such that $U$ and $W$ are orthogonal subspaces. In other words, if $U = \langle (x,y,z) \rangle$ and $W = \langle (u,v,w) \rangle$ then $xu + yv + zw = 0$. This indeed defines a graph since the equation $xu + yv + zw = 0$ is symmetric, and these graphs are called Erdős-Rényi polarity graphs. It is straightforward to check that $ER_q$ does not have any quadrilaterals, although it contains triangles. Also, $ER_q$ has exactly $q^2 + q + 1$ vertices and every one-dimensional subspace in $[\frac{V}{1}]$ that is not orthogonal to itself has degree $q + 1$ in the graph. A counting argument shows that there are exactly $q + 1$ self-orthogonal subspaces, which as vertices each have degree $q$, and so

$$e(ER_q) = \frac{1}{2}(q + 1)q + \frac{1}{2}q^2(q + 1) = \frac{1}{2}q(q + 1)^2.$$ 

Once more the distribution of primes shows $\text{ex}(n, C_4) \gtrsim \frac{1}{2}n^{3/2}$. The following deep theorem of Füredi (1988) shows that $\text{ex}(n, C_4) = e(ER_q)$ when $n = q^2 + q + 1$ and $q$ is a prime power:

**Theorem 2.** Let $G$ be an extremal quadrilateral-free graph with $n = q^2 + q + 1$ vertices where $q$ is a prime power. Then $e(G) \leq \frac{1}{2}q(q + 1)^2$.

The main difficulty is in improving the upper bound $d(d-1) \leq n-1$ for a $C_4$-free $n$-vertex graph of average degree $d$, since when $n = q^2 + q + 1$ this only gives $\text{ex}(n, C_4) \leq \frac{1}{2}(q + 1)n$. We remark that the graphs $ER_q$ may be described geometrically by polarities of projective planes – see Lazebnik, Ustimenko and Woldar for details on polarity graphs.

### 4.5 Dependent random choice

If $F$ is a family of graphs containing a bipartite graph, then for some $\delta > 0$, $\text{ex}(n,F) < n^{2-\delta}$ by the Kövari-Sós-Turán Theorem. Erdős and Simonovits conjectured that for every finite family graph $F$ of graphs, there exists $\alpha = \alpha(F)$ such that $\text{ex}(n,F) = \Theta(n^\alpha)$. This $\alpha$ is called the exponent of $F$. If $F$ is a bipartite graph containing a cycle, then in general it is not known if $F = \{F\}$ has an exponent, even for simple graphs such as $F = K_{4,4}$ and $F = C_8$. In these notes, we give an approach to finding upper bounds on the exponent using a method called Dependent Random Choice, which has many other applications besides. We refer the reader to Fox and Sudakov for a survey of dependent random choice.
The method of dependent random choice was developed from early ideas of Gowers and Kostochka and Rödl, and is implicit in works of Füredi. If \( G = (V,E) \) is an \( n \)-vertex graph with average degree \( d \), then on average a sequence \( R \in V^r \) has \( d^r/n^{r-1} \) common neighbors. However, there may be many sequences \( R \) which have no common neighbors, for instance if \( G \) has a small dense component plus many isolated vertices. The idea of dependent random choice is to locate a large set \( Z \subset V \) such that every element of \( Z^r \) has many common neighbors, by letting \( Z \) be the common neighborhood of a random sequence of vertices. This has the effect of biasing the elements of \( Z^r \) to have many common neighbors as oppose to a randomly chosen set \( Z \) of the same size.

The following notation will be used. Throughout, \( a,b,m,r,s,t,n \) are positive integers. Let \( G = (V,E) \) be a graph and \( R \in V^r \) i.e. \( R \) is a sequence of \( r \) vertices of \( V \) with replacement. Then \( N(R) = \bigcap_{v \in R} N(v) \) and \( d(R) = |N(R)| \). We say \( R \) is \( b \)-rich if \( d(R) \geq b \) and \( b \)-poor otherwise. A set \( Z \subset V \) is \((r,b)\)-rich if every sequence in \( Z^r \) is \( b \)-rich. The next theorem gives a density condition on a graph which guarantees a large \((r,b)\)-rich subset of vertices:

**Theorem 4.11.** Let \( G = (V,E) \) be an \( n \)-vertex graph of average degree \( d \), where

\[
d^a n^{1-a} - (b-1)^a n^{r-a} > m - 1. \tag{4.5.1}
\]

Then there exists \( Z \in \binom{V}{m} \) such that \( Z^r \) is \((r,b)\)-rich.

**Proof.** Randomly select a sequence \( S \in V^a \), and let \( W \) be the number of \( b \)-poor sequences in \( N(S)^r \). Let \( \mathcal{R} \) be the set of \( b \)-poor sequences in \( V^r \). Then

\[
\mathbb{E}(W) = \frac{1}{n^a} \sum_{R \in \mathcal{R}} d(R)^a \leq (b-1)^a n^{r-a}.
\]

Delete one vertex from each \( b \)-poor sequence in \( N(S)^r \) to get a set \( Z \subset N(S) \). Then by convexity and (4.5.1),

\[
\mathbb{E}(|Z|) \geq \frac{1}{n^a} \sum_{S \in V^a} d(S) - (b-1)^a n^{r-a} = \frac{1}{n^a} \sum_{v \in V} d(v)^a - (b-1)^a n^{r-a} \geq d^a n^{1-a} - (b-1)^a n^{r-a} > m - 1.
\]

Therefore there exists a choice of \( S \) so that \(|Z| \geq m\). \( \square \)

### 4.6 Embedding bipartite graphs

Theorem 4.11 allows the embedding of bipartite graphs \( F \) with parts \( X \) and \( Y \) of size \( s \) and \( t \), where all vertices in \( Y \) have degree at most \( r \). We use the following elementary embedding lemma:

**Lemma 4.12.** Let \( F \) be a bipartite graph with parts \( X \) and \( Y \) such that \(|X| = s\) and \(|Y| = t\) and every vertex of \( Y \) has degree at most \( r \). If \( G = (V,E) \) is a graph and \( Z \in \binom{V}{s} \) is \((r,t+s-r)\)-rich, then \( F \subset G \).
Proof. Bijectively map the vertices of $F$ in $X$ to the vertices of $Z$. Now suppose we have embedded vertices $y_1, y_2, \ldots, y_u$ of $Y$ in $V(G) \setminus Z$, where $u < t$. Then $y_{u+1}$ has at most $r$ neighbors amongst $y_1, y_2, \ldots, y_u$, which we may assume are $y_1, y_2, \ldots, y_q$. Then the sequence $(y_1, y_2, \ldots, y_r)$ is $(t+s-r)$-rich, so $(y_1, y_2, \ldots, y_r)$ has at least $t$ common neighbors in $V(G) \setminus Z$. Since $u < t$, there exists a vertex $v \in V(G) \setminus Z$ adjacent to $y_1, y_2, \ldots, y_r$, and now we may map $y_{u+1}$ to $v$. \qed

**Theorem 4.13.** Let $F$ be a bipartite graph with parts $X$ and $Y$ such that $|X| = s$ and $|Y| = t$ and every vertex of $Y$ has degree at most $r$. Then for any integer $a \geq r$,

$$\text{ex}(n, F) \leq \frac{1}{2} (s-1) \frac{s}{s-a} n^{2-\frac{1}{a}} + \frac{1}{2} (t-s-r-1)n^{1+\frac{r}{a}}. \quad (4.6.1)$$

In particular, $\text{ex}(n, F) = O(n^{2-\frac{1}{r}})$.

**Proof.** If $G$ is an $n$-vertex $F$-free graph with average degree $d$, then by the embedding lemma and (4.5.1) with $m = s$ and $b = t + s - r$,

$$d^a n^{1-a} - (t + s - r - 1)^a n^{r-a} \leq s - 1.$$

This gives (4.6.1). Taking $a = r$ we obtain $\text{ex}(n, F) = O(n^{2-\frac{1}{r}})$. \qed

### 4.7 Subdivisions

A subdivision of a graph $G$, denoted $S(G)$, is a graph obtained by replacing each edge with a path of length two with the same ends as the edge, such that all the paths are internally disjoint. For a graph $n$ vertices and $m$ edges, the subdivision is a bipartite graph with parts of size $m$ and $n$, and every vertex in the part of size $m$ has degree two. The following is proved using Theorem 4.13 with a careful choice of $a$.

**Theorem 4.14.** If $G$ has $n$ vertices and $pm^2$ edges then $S(K_{[p,\sqrt{m}]} \subset G$.

**Proof.** Let $s = [p,\sqrt{m}]$. By (4.5.1) with $r = 2$ and $b = s + \binom{s}{2}$, a calculation gives:

$$\text{ex}(n, S(K_s)) < \frac{1}{2} p^2 n^{2-\frac{1}{2}} + \frac{1}{2} n^{2+\frac{1}{2}}$$

for any integer $a \geq 2$. Select $a = [\log_{1/p^2} n]$. Then

$$\text{ex}(n, S(K_s)) < \frac{1}{2} pn^2 + \frac{1}{2} pn^2 < pn^2.$$ 

So if $G$ is a graph with at least $pm^2$ edges then $S(K_s) \subset G$ as required. \qed

One cannot expect an $S(K_m)$ in a graph if $\binom{m}{2} + m > n$, and so the bound of order $n^{\frac{1}{2}}$ on $m$ in the above theorem is almost tight. With an additional idea, Alon, Krivelevich and Sudakov showed that one can find $S(K_{[p,\sqrt{m}]}$ in the above theorem, which is tight in the dependence on $p$.  

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8
4.8 Random graphs and random polynomial graphs

The explicit constructions of graphs without specific subgraphs often involve adjacency defined by systems of polynomial equations over finite fields. This motivates the rational exponent conjectures of Erdős and Simonovits (1982). The first conjecture is wide open:

**Conjecture 4.15. (Exponent conjecture)** For every family $\mathcal{F}$ consisting of finitely many bipartite graphs containing cycles, there exist constants $\alpha \in [1,2)$ and $\beta > 0$ such that $\text{ex}(n, \mathcal{F}) = \beta n^\alpha + o(n^\alpha)$ as $n \to \infty$.

In the case of $C_4$, we recall $\text{ex}(n, C_4) \sim \frac{1}{2} n^{3/2}$ so the exponent is $3/2$. In general, however, the exponent conjecture is open, even for such simple graphs as $C_6$ and $K_{4,4}$. A second conjecture was made by Erdős and Simonovits (1982):

**Conjecture 4.16. (Rational exponent conjecture)** For every rational $\alpha \in [1,2)$ there exists a bipartite graph $F$ such that $\text{ex}(n, F) = \Theta(n^\alpha)$.

The purpose of these notes is to prove that at least this conjecture is true if we replace $F$ with a family $\mathcal{F}$ of bipartite graphs, via randomized algebraic constructions, based on the ideas of Bukh and Conlon (2014). We first discuss the construction of dense $F$-free graphs via the Erdős-Rényi model of random graphs, which generally does not even give the order of magnitude of extremal numbers.

4.9 Random graphs

Recall the **Erdős-Rényi random graph**, $G_{n,p}$, denotes the discrete probability space whose sample space is the set $\Omega_n$ of all graphs on $[n]$ with product probability measure

$$
P(G) = p^{e(G)}(1 - p)^{|\Omega_n| - e(G)}
$$

and $\sigma$-field $2^{\Omega_n}$. This model was introduced by Erdős and Rényi (1961), and is also known as the mean field model by statistical physicists. The book by Bollobás (2001) is a standard reference for random graphs. Random graphs provide simple constructions of dense $F$-free graphs. Indeed if $F$ is any graph, and $X_F(G)$ is the number of copies of $F$ in a graph $G$, then

$$
E(X_F) = |\text{aut}(F)|p^{e(F)}\left(\frac{n}{v(F)}\right)^{n^{e(F)}} < p^{e(F)}n^{v(F)}.
$$

If $Y$ is the number of edges in the random graph, then $E(Y) = p\binom{n}{2}$. It follows by deleting an edge from every copy of $F$ that there exists an $n$-vertex $F$-free graph $G^*$ with at least $p\binom{n}{2} - N$ edges. This gives the following theorem:

**Theorem 4.17.** For any graph $F$ with $e(F) \geq 2$, there exists an $n$-vertex $F$-free graph $G^*$ with

$$
e(G^*) \geq 4^{-\frac{1}{e(F)-1}}n^{2-\frac{e(F)-2}{e(F)-1}}.
$$

(4.9.1)

**Proof.** We maximize $p\binom{n}{2} - p^{e(F)}n^{v(F)}$ over $p \in [0,1]$, since this is a lower bound on the expected
number of edges in the graph $G^*$. Selecting $p^{v(F)-1} = \frac{1}{4}n^{2-v(F)}$ gives \((4.9.1)\). □

In general, the it is unlikely that \((4.9.1)\) is ever tight up to constant factors, unless $F$ is a tree. Certainly the bound is never tight for graphs of chromatic number at least three, since they have quadratic Turán Numbers. If $F = C_4$, then the above theorem only gives $\text{ex}(n, F) = \Omega(n^{4/3})$, whereas we know $\text{ex}(n, F) = \Theta(n^{3/2})$. It turns out one can often do better than random graphs by appealing to random polynomials.

4.10 Random polynomials

Let $d, t \in \mathbb{Z}^+$ and let $q$ be a prime power. For a vector $w \in \{0, 1, \ldots, q-1\}^t$, let $|w| = \sum_{i=1}^{t} w_i$. Let $Z_w : w \in \{0, 1, \ldots, q-1\}^t$ be independent uniform random variables with values in $\mathbb{F}_q$. A random polynomial of degree at most $d$ in $t$ variables $X_1, X_2, \ldots, X_t$ over $\mathbb{F}_q$ is defined by

$$f(X_1, X_2, \ldots, X_t) = \sum_{|w| \leq d} Z_w \prod_{i=1}^{t} X_i^{w_i}.$$ 

In other words, each monomial of degree at most $d$ has a random coefficient from $\mathbb{F}_q$. Throughout we use $f$ to denote a random polynomial of degree $d$, and let $\Omega(q, t, d)$ be the sample space of all polynomials of degree at most $d$ in $t$ variables over $\mathbb{F}_q$. Note that for a fixed polynomial $f_0 \in \Omega(q, t, d)$ we have the probability measure $P(f = f_0) = \frac{1}{q^T}$ where $T = \frac{d+1}{t} \binom{d+t}{t-1}$. The first easy lemma is as follows:

**Lemma 4.18.** For any $x \in \mathbb{F}_q^t$,

$$P(f(x) = 0) = \frac{1}{q}.$$ 

We require a generalization of this lemma to families of polynomial equations. If $S \subset \mathbb{F}_q^t$, then clearly the events $\{f(x) = 0\}$ for $x \in S$ need not be stochastically independent, if $|S|$ is too large relative to $d$ or $q$. The following lemma, due in a different form to Bukh (2011), quantifies this statement:

**Lemma 4.19.** If $S \subset \mathbb{F}_q^t$ has size $r$ and $\min\{d, q-1\} \geq r-1$, then

$$P(\forall x \in S : f(x) = 0) = \frac{1}{q^r}.$$ 

**Proof.** Let $S = \{x_1, x_2, \ldots, x_r\}$ and denote by $x_{ij}$ the $j$th co-ordinate of $x_i$. Consider $m$ equations $f(x_i) = 0$ as equations in the $T = \frac{d+1}{t} \binom{d+t}{t-1}$ coefficients $f$. Since $d \geq r-1$, the coefficient matrix contains the block matrix $V = (V(1), V(2), \ldots, V(t))$ where $V(j)$ is the $r \times r$ Vandermonde
matrix defined by:

\[
V(j) = \begin{pmatrix}
1 & x_{1j} & x_{1j}^2 & \cdots & x_{1j}^{r-1} \\
1 & x_{2j} & x_{2j}^2 & \cdots & x_{2j}^{r-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & x_{rj} & x_{rj}^2 & \cdots & x_{rj}^{r-1}
\end{pmatrix}.
\]

As is well-known, when \( q \geq r \), \( V(j) \) has full rank unless two of its rows are identical. The reader may verify that the same holds for the whole matrix: since no two rows of \( V \) are equal, \( \text{rank}(V) = r \). Then \( \text{nullity}(V) = T - r \), which implies the lemma.

If \( x \in \mathbb{F}_q^t \) and \( y \in \mathbb{F}_q^t \), we write \((x, y)\) for their concatenation. Throughout these notes, independence of polynomials refers to stochastic independence of random polynomials.

### 4.11 Random polynomial graphs

For any \( S = \{f_1, f_2, \ldots, f_r\} \subset \Omega(q, 2t, d) \), let \( \mathcal{G}_S \) be the bipartite graph whose parts are \( A = B = \mathbb{F}_q^t \) and where \( x \in A \) and \( y \in B \) form an edge if \( f_i(x, y) = 0 \) for \( i = 1, 2, \ldots, r \). Let \( \mathcal{G} = \mathcal{G}(r, t, d) \) be the random graph obtained when \( f_1, f_2, \ldots, f_r \) are independent random polynomials of degree at most \( d \). For a bipartite graph \( F \), let \( X_F(\mathcal{G}) \) be the number of copies of \( F \) in \( \mathcal{G} \).

**Lemma 4.20.** Let \( r, t, d \) be positive integers with \( t > r \), and \( \mathcal{G} = \mathcal{G}(r, t, d) \), and let \( F \) be a bipartite graph with \( e(F) \leq \min\{q, d + 1\} \). Then

\[
E(X_F(\mathcal{G})) \leq q^{tv(F) - re(F)}.
\]

In particular, \( E(e(\mathcal{G})) = q^{2t - r} \).

**Proof.** Let \( m = e(F) \). A fixed copy of \( F \) on \( V(\mathcal{G}) \) appears in \( \mathcal{G} \) with probability

\[
P(\forall \{x, y\} \in E(F), x \in A, y \in B : f(x, y) = 0)^r
\]

since \( \mathcal{G} \) is generated by \( r \) random independent polynomials. By Lemma 4.19, since \( \min\{d, q - 1\} \geq m - 1 \), the above probability is \( q^{-re(F)} \), which implies the lemma.

We conclude that with positive probability, \( \mathcal{G} \) has an \( F \)-free subgraph \( H \) where

\[
e(H) \geq E(e(\mathcal{G}) - X_F(\mathcal{G})) \geq q^{2t - r} - q^{tv(F) - re(F)}
\]

Unfortunately, this offers only a lower bound on \( \text{ex}(n, F) \) of the same order of magnitude as in Theorem 4.17. To improve on this, the idea is that for many bipartite graphs \( F^* \), the copies of \( F^* \) in \( \mathcal{G} \) are not uniformly distributed, due to estimates on the number of points on affine varieties.

Let \( \overline{\mathbb{F}}_q \) denote the algebraic closure of \( \mathbb{F}_q \), and let \( f_1, f_2, \ldots, f_r \in \mathbb{F}_q[X_1, X_2, \ldots, X_t] \). An **affine algebraic variety** defined over \( \mathbb{F}_q \) is a set \( V \subset \overline{\mathbb{F}}_q^t \) of the form

\[
V = \mathcal{V}(f_1, f_2, \ldots, f_r) = \{x \in \overline{\mathbb{F}}_q^t : \forall i, f_i(x) = 0\}.
\]
We say that \( \mathcal{V} \) is generated by \( f_1, f_2, \ldots, f_r \). The dimension \( \dim(\mathcal{V}) \) of a variety is the length \( k \) of a maximal chain of subvarieties \( \mathcal{V}_0 \subseteq \mathcal{V}_1 \subseteq \cdots \subseteq \mathcal{V}_k = \mathcal{V} \). The book by Mumford is a good reference on varieties.

The complexity of a variety \( \mathcal{V} \subseteq F_q^t \) generated by polynomials \( f_1, f_2, \ldots, f_r \) with \( \max \{ \deg(f_i) : 1 \leq i \leq r \} = d = \max\{r, t, d\} \). An \( F_q \)-variety is any variety fixed under the Fröbenius automorphism \( x \mapsto x^q \). A variety \( \mathcal{V} \) is irreducible if it is not a union of any proper subvarieties. We write \( |\mathcal{V}_q| \) for the number of points of \( \mathcal{V}_q \) whose co-ordinates are all in \( F_q \). The key fact for us is the following, which follows from a well-known result called the Lang-Weil Bound:

**Lemma 4.21.** If \( \mathcal{V}_q \) has complexity \( b \), then there exist \( c, t \in \mathbb{Z}^+ \cup \{0\} \) such that as \( q \to \infty \):

\[ |\mathcal{V}_q| = (c + O_b(q^{-1/2}))q^{\dim(\mathcal{V}_q)}. \]

Here the value of \( c \) is known explicitly: it is the number of geometrically irreducible components of \( \mathcal{V} \) of the same dimension as \( \mathcal{V} \) – these are components fixed by the Fröbenius automorphism – and in particular \( c = 1 \) if \( \mathcal{V} \) is irreducible. The Lang-Weil bound says, in particular, that given a system of \( r \) polynomial equations in \( t \) variables over \( F_q \) of degree at most \( d \), then as \( q \to \infty \) the number of solutions in \( F_q \) is either a constant bounded by a function of \( b \) or unbounded as a function of \( q \).

The following lemma is due to Bukh (2011):

**Lemma 4.22.** For every \( b \in \mathbb{Z}^+ \), there exist \( c_1, c_2 \) such that for every variety \( \mathcal{V}_q \) defined over \( F_q \) with complexity at most \( b \), \( \mathcal{V}_q \) is a union of a set of at most \( c_1 \) irreducible varieties of complexity at most \( c_2 \), each defined over \( F_q \).

The final lemma deals with intersection of varieties.

**Lemma 4.23.** If \( \mathcal{V}_q \subseteq \overline{F}_q^t \) is an irreducible variety over \( F_q \) with positive dimension, and \( f \in \overline{F}_q[X_1, X_2, \ldots, X_t] \), then \( \mathcal{V} \subseteq \mathcal{V}(f) \) or \( \dim(\mathcal{V} \cap \mathcal{V}(f)) < \dim(\mathcal{V}) \).

Recall that if \( F \) and \( G \) are graphs, then a homomorphism from \( F \) into \( G \) is a map \( \phi : V(F) \to V(G) \) such that \( \{u, v\} \in E(F) \) implies \( \{\phi(u), \phi(v)\} \in E(G) \). In other words, the map is edge-preserving (but may map non-adjacent vertices onto each other). Let \( S \subseteq \overline{F}_q^t \) and consider the graph \( \mathcal{G}_S \). If \( F \) is a graph on \( [k] \), and \( F \) is a subgraph of \( \mathcal{G}_S \), then there exists \( x \in \overline{F}_q^k \) such that \( x = (x_1, x_2, \ldots, x_k) \) and for all \( i \in S \) and all \( \{i, j\} \in F \), \( f(x_i, x_j) = 0 \). If we fix a set \( L = \{1, 2, \ldots, \ell\} \subseteq V(F) \) corresponding to \( M = \{x_1, x_2, \ldots, x_\ell\} \subseteq V(\mathcal{G}) \), then define the variety

\[ \mathcal{V} = \mathcal{V}(F, L) = \{ (x_{\ell+1}, x_{\ell+2}, \ldots, x_k) \in \overline{F}_q^{(k-\ell)} | \forall \{i, j\} \in F, \forall f \in S : f(x_i, x_j) = 0 \}. \]

Let \( H \) be a graph such that there exists a map \( \phi : V(F) \to V(H) \) that preserves edges of \( F \) and preserves the vertices in \( L \). Thus \( \phi \) is a homomorphism of \( F \) preserving \( L \). Then there is a one-to-one correspondence between the points \( x \in \mathcal{V} \cap \overline{F}_q^{(k-\ell)} \) and the homomorphisms \( \phi : V(F) \to V(\mathcal{G}) \) preserving \( L \), namely given \( x = (x_{\ell+1}, x_{\ell+2}, \ldots, x_k) \), define the vertex set of \( H \) to be \( \{x_i : i \in [k]\} \) and define the homomorphism \( \phi_x \) by \( \phi_x(i) = x_i \) for \( i \in [k] \), so that

\[ E(H) = \{ (\phi_x(i), \phi_x(j)) : \{i, j\} \in E(F) \}. \]
For a homomorphism $\phi : V(F) \to V(G)$ of $F$ preserving $L$, define
\[ V_\phi = \{ x \in V : \phi x = \phi \}. \]
We are going to count points in $V \cap \mathbb{F}_q^{d(k-\ell)}$ which correspond to isomorphisms of $F$ in $\mathcal{G}$ containing $M$.

If $\Phi$ is the set of these homomorphisms, then we may write $V = \bigcup_{\phi \in \Phi} V_\phi$. We call $\phi \in \Phi$ a proper homomorphism if it is not an isomorphism of $F$. Let $\Phi^*$ be the set of isomorphisms in $\Phi$. We may also write $V = V_1 \cup V_2 \cup \cdots \cup V_\sigma$ where $V_i$ are irreducible varieties of bounded complexity using Lemma 4.22. We observe:

- If $\dim(V_i) = 0$, then by the Lang-Weil Bound, $|V_i| \leq c$.
- If $V_i \subseteq V_\phi$ for some $\phi \notin \Phi^*$, then $V_i$ does not contribute any isomorphic copies of $F$ containing $L$.

If the number of isomorphic copies of $F$ in $\mathcal{G}_S$ containing $M$ exceeds $c\sigma$, then there exists $i$ such that $\dim(V_i) \geq 1$ and $V_i \subseteq V_\phi$ for all $\phi \notin \Phi^*$. We conclude by Lemma 4.23 that $\dim(V_i \cap V_\phi) < \dim(V_i)$ for all $\phi \notin \Phi^*$. Let
\[ V^* = \bigcup_{\phi \in \Phi^*} V_\phi. \]
In other words $|V^*|$ counts the number of copies of $F$ containing $M$ in $\mathcal{G}_S$. Then by the Lang-Weil Bound applied to $V_i \cap V_\phi$,
\[
|V^*| \geq |V_i| - \sum_{\phi \in \Phi \setminus \Phi^*} |V_i \cap V_\phi| \\
\geq (1 + O(q^{-1/2})q^{\dim(V_i)}) - O\left( \sum_{\phi \in \Phi \setminus \Phi^*} q^{\dim(V_i) - 1} \right) \geq q.
\]
This is summarised by the following lemma:

**Lemma 4.24.** For all $r, t, d \in \mathbb{Z}^+$, there exists $s_0$ such that if $S \subseteq \Omega(q, 2t, d)$ and $F$ is a bipartite graph and $L \subseteq V(F)$, and $M$ is a set of $|L|$ vertices in $\mathcal{G}_S$, then $\mathcal{G}_S$ contains at most $s_0$ copies of $F$ containing $M$ or at least $q - o(q)$ copies of $F$ containing $M$.

Fixing a bipartite graph $F$ and $L \subseteq V(F)$, let $F_s(L)$ denote the set of bipartite graphs of the form $F_1 \cup F_2 \cup \cdots \cup F_s$ where each $F_i$ is isomorphic to $F$, the $F_i$ are all distinct, and all share the set $L$. We say that $F$ is $L$-balanced if for any $F^* \in \bigcup_{s \geq 1} F_s(L)$, we have
\[
\frac{e(F^*) - e(L)}{e(F) - e(L)} \geq \frac{v(F^*) - v(L)}{v(F) - v(L)}.
\]

**Theorem 4.25.** Let $F$ be an $L$-balanced bipartite graph. Then there exists $s_1 = s_1(F)$ such that for $s \geq s_1$,
\[
x(n, F_s(L)) \geq n^{2 - \frac{e(F) - |L|}{e(F) - e(L)}}.
\]

**Proof.** Let $v(F) = k$, $|L| = \ell$ and $e(F) - e(L) = m$. Fix $\mathbb{F}_q$, and consider the random polynomial
graph \( \mathcal{G} = \mathcal{G}(r, t, d) \) where \( rm = (k - \ell)t \). Let \( s_0 \) be the constant returned by Lemma \( 4.24 \). Fix the image \( M \) of \( L \) in \( V(\mathcal{G}) \). If \( s \geq s_0 \) and \( F^* \in \mathcal{F}_s(M) \) is contained in \( \mathcal{G} \), then by Lemma \( 4.24 \) there exist \( \left( q^{o(q)} \right) \) copies of \( F^* \) in \( \mathcal{G} \) containing \( M \). Let \( s_1 = \max \{ s_0, \ell t + 1 \} \) and let \( s \geq s_1 \) satisfy \( sm \leq \min\{ d + 1, q \} \). Let \( X_M \) count the number of \( F^* \in \mathcal{F}_s(M) \) which are contained in \( \mathcal{G} \) and contain \( M \). Then by Markov’s Inequality,

\[
P(X_M \geq 1) = P(X_M \geq \left( q^{o(q)} \right)) \leq \frac{E(X_M)}{\left( q^{o(q)} \right)}.
\]

Since \( e(F^*) \leq \min\{ d + 1, q \} \), Lemma \( 4.20 \) gives:

\[
E(X_M) \leq q^{-r(e(F^*) - e(L))} \cdot q^{t(v(F^*) - \ell)}.
\]

Since \( F \) is \( L \)-balanced and \( rm = (k - \ell)t \), \( r(e(F^*) - e(L)) \leq t(v(F^*) - \ell) \) which implies

\[
E(X_M) \leq 1.
\]

Let \( X^* \) be the total number of subgraphs of \( \mathcal{G} \) which are isomorphic to a graph in \( \mathcal{F}_s(L) \). Then

\[
E(X^*) \leq \sum_{M \subset V(\mathcal{G})} E(X_M) \left( \frac{q^{o(q)}}{s} \right)^{-1} \leq s^s |V(\mathcal{G})|^\ell q^{-s} \leq 2^\ell s^s q^{\ell t - s}.
\]

Since \( s > \ell t \), this is less than 1 for large enough \( q \), and so for large enough \( q \),

\[
E(e(\mathcal{G}) - X^*) > q^{2t - r} - 1.
\]

So with positive probability and for large enough \( q \), \( \mathcal{G} \) is \( \mathcal{F}_s(L) \)-free and has at least \( q^{2t - r} = q^{2 - \frac{k - \ell}{m}} \) edges. Using the distribution of primes, we conclude \( \text{ex}(n, \mathcal{F}_s(T)) \geq q^{2t - r} \), as required.

The following theorem was proved by Conlon (2015).

**Corollary 4.26.** For every \( k \) there exists \( s \) such that \( \text{ex}(n, \theta_{k,s}) = \Theta(n^{1 + \frac{1}{k}}) \) and \( \text{ex}(n, K_{k,s}) = \Theta(n^{2 - \frac{1}{k}}) \).

**Proof.** Faudree and Simonovits (1983) showed \( \text{ex}(n, \theta_{k,s}) = O(n^{1 + \frac{1}{k}}) \) using breadth-first search tree arguments as in the Even Cycle Theorem. For the lower bound, we apply the last theorem with \( F \) a path of length \( k \), and \( L \) the ends of \( F \). Then it can be verified that \( F \) is \( L \)-balanced, and then the last theorem gives

\[
\text{ex}(n, \theta_{k,s}) \geq \text{ex}(n, \mathcal{F}_s(L)) \geq n^{1 + \frac{1}{k}}.
\]

This completes the proof for \( \theta_{k,s} \). For \( K_{k,s} \), we apply the same procedure with \( F = K_{1,k} \) and \( L \) the set of leaves of \( F \) – note \( F \) is clearly \( L \)-balanced since for all \( s, \mathcal{F}_s(L) = \{ K_{k,s} \} \).

**Corollary 4.27.** If \( F \) is the bipartite graph with vertex set \( \{ u_i, v_i : 1 \leq i \leq 2 \} \cup \{ w_1, x_1, w_2, x_2 \} \) and \( E(F) = \{ \{ u_i, v_i \}, \{ u_i, w_j \}, \{ v_i, x_j \} : 1 \leq i, j \leq 2 \} \) and \( L = \{ w_1, x_1, w_2, x_2 \} \), then if \( s \) is large enough, \( \text{ex}(n, \mathcal{F}_s(L)) = \Theta(n^{s^2/5}) \).

**Proof.** Every vertex of \( F \) not in \( L \) has degree exactly three. Therefore \( F \) is balanced, so by Theorem
if \(s \geq s_0\) then
\[
\text{ex}(n, \mathcal{F}_s(L)) \gtrsim n^{8/5}.
\]

On the other hand, if \(G\) is an \(n\)-vertex graph of average degree \(d \geq 4\), then \(G\) contains \(\Omega(d^5) \cdot n\) copies of the tree \(F\). If this exceeds \(sn^4\), then there exists a set of four vertices in \(G\) which are the image of \(L\) in at least \(s\) copies of \(F\) in \(G\), and their union forms a graph in \(\mathcal{F}_s(L)\). So if \(G\) is \(\mathcal{F}_s(L)\)-free, then \(d^5 = O(n^3)\) which gives \(e(G) = O(n^{8/5})\), as required.

If \(G\) is a graph and \(R \subset V(G)\), let \(\partial R\) denote the set of edges with at least one end in \(R\). Let \(F\) be a graph and \(L \subset V(F)\) an independent set. Suppose that every \(v \in U = V(F) \setminus L\) has \(i\) neighbors in \(L\). Then the statement that \(F\) is \(L\)-balanced is equivalent to the statement
\[
\frac{|\partial R|}{|R|} \geq \frac{|\partial U|}{|U|}
\]
for every \(R \subset U\) in the graph \(F - L\).

Intuitively, this method should be able to prove Conjecture 2: for every rational \(\alpha \in (1, 2)\) there exists a bipartite graph with exponent \(\alpha\). The main technical difficulty however is finding an \(L\)-balanced bipartite graph \(F\) such that
\[
\text{ex}(n, F) = O(n^{2 - \frac{v(F) - e(L)}{\alpha(F) - e(L)}})
\]
to match the lower bound in Theorem 4.25. For instance, suppose \(T\) is the tree with nine vertices and height two such that every vertex other than the six leaves has degree three. Then let \(L\) be the set of leaves of \(T\), and let \(F\) be created by taking \(s \geq 3\) copies of \(T\) pairwise intersecting only in \(L\). In other words, take trees \(T_1, T_2, \ldots, T_s\) each isomorphic to \(T\), with \(V(T_i) \cap V(T_j) = L\) for \(i \neq j\). Then it seems likely that \(\text{ex}(n, F) = O_s(n^{14/9})\): we know the number of isomorphic copies of \(T\) in a graph with \(\Theta_s(n^{14/9})\) edges is \(\Omega_s(n^6)\) (see the notes on the Moore bound), and therefore one should expect many trees to share \(L\) but otherwise be disjoint. Unfortunately these trees may intersect in non-leaves, which complicates the problem. Note that \(\text{ex}(n, F) = \Omega(n^{14/9})\) due to the randomized polynomial construction – the details are left to the reader. By taking \(\mathcal{F}_s(L)\) for appropriate \(s\) and \(L\), one can obtain the following by generalizing Corollary 2:

**Theorem 4.28.** For every \(\alpha \in (1, 2)\), there exists a family \(\mathcal{F}\) of bipartite graphs such that\n\[
\text{ex}(n, \mathcal{F}) = \Theta(n^\alpha).
\]