# Examples – Connectivity

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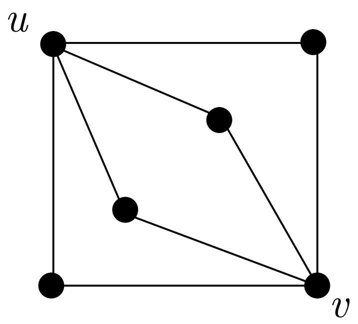
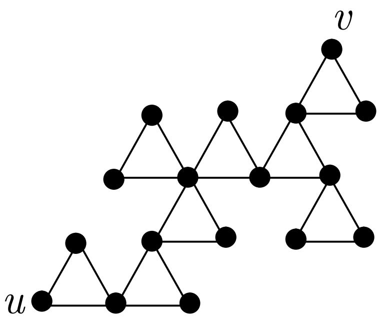
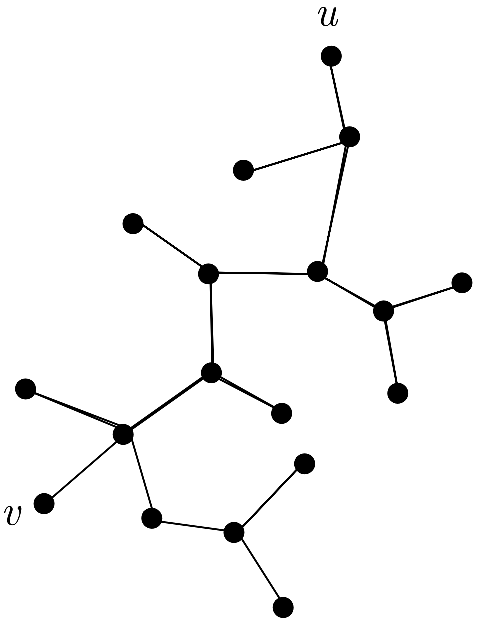
**Example 1.** For each of the graphs below, find

$$\kappa(u,v),\ \lambda(u,v),\ \kappa(G),\ \lambda(G),\ \delta(G)$$

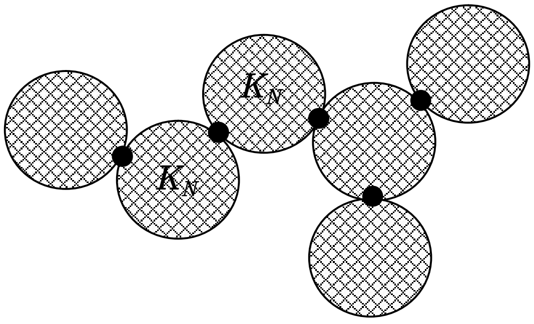




Tree of triangles



Tree



Tree of $K_N$

**Solution.**

For the first graph, $u$ and $v$ can be separated by removing all other vertices in the graph. Therefore $\kappa(u, v) \leq 4$.

On the other hand, $\kappa(u, v)$ equals the maximum number of internally disjoint $uv$ paths by Menger's Theorem. Since we can find four internally disjoint $uv$ paths in the graph, $\kappa(u, v) \geq 4$.

We conclude $\kappa(u, v) = 4$.

Since it is always true that $\lambda(u, v) \geq \kappa(u, v)$ we get $\lambda(u, v) \geq 4$. Clearly removing all edges on $u$ separates $u$ from $v$, so $\lambda(u, v) \leq 4$.

We conclude $\lambda(u, v) = 4$.

Recall $\kappa(G) \leq \lambda(G) \leq \delta(G)$. Since $\delta(G) = 2$ we get $\kappa(G) \leq \lambda(G) \leq 2$. By inspection, $\kappa(G) = \min \kappa(x, y) = 2$ (the bottom left and top right vertices can be separated by removing $u$ and $v$, and by inspection this is the smallest set of vertices which disconnects the graph).

We conclude $\kappa(G) = \lambda(G) = \delta(G) = 2$

For the second graph, the tree of triangles, $u$ and $v$ can be separated by removing one vertex in the graph (many choices). Therefore $\kappa(u, v) \leq 1$. Since the graph is connected, $\kappa(u, v) \geq 1$.

We conclude $\kappa(u, v) = 1$.

$u$ and $v$ can be separated by removing two edges, say the edges on $u$, so $\lambda(u, v) \leq 2$. Now there are two edge-disjoint $uv$ paths, so by Menger's Theorem that means $\lambda(u, v) \geq 2$.

We conclude $\lambda(u, v) = 2$.

Since G is connected, $\kappa(G) \geq 1$.

We conclude $\kappa(G) = \min \kappa(u, v) = 1$

Since every edge of G is in a cycle, there can be no bridges in the graph. That means $\lambda(G) \geq 2$. But also $\lambda(G) = \min \lambda(x, y) \leq 2$.

We conclude $\lambda(G) = \min \lambda(u, v) = 2$

Also $\delta(G) = 2$ by inspection.

For the tree, since every edge is a bridge we have for any $u$,$v$ not adjacent,

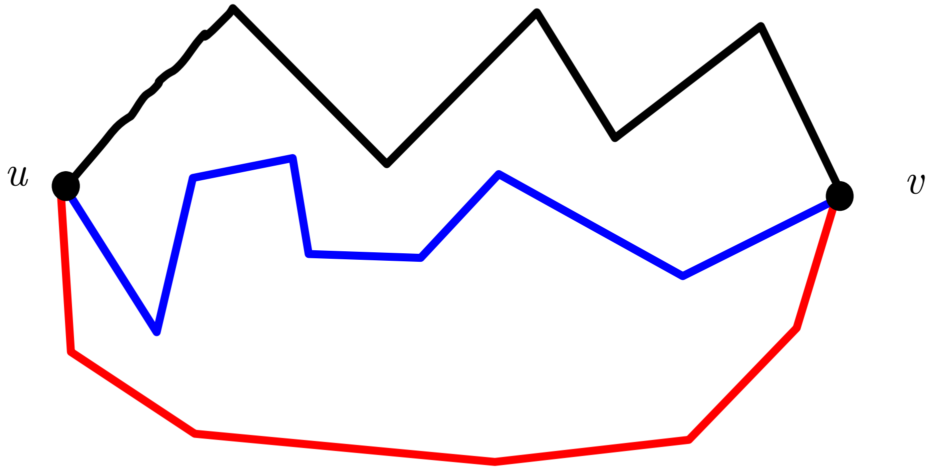
$$\lambda(G) = \kappa(G) = \delta(G) = \kappa(u, v) = \lambda(u, v) = 1$$

For the tree of $K_N$, notice that any two vertices are joined by $N - 1$ edge-disjoint paths, so by Menger's Theorem, $\lambda(G) = \min \lambda(u, v) = N - 1$. Clearly for any u,v non-adjacent, $\kappa(G) = \kappa(u, v) = 1$. Finally, $\delta(G) = N - 1$.

**Example 2.** Show that every 3-connected graph has an even cycle.

**Solution.**

By Menger's Theorem, any two non-adjacent vertices $u$,$v$ in a 3-connected graph are joined by 3 internally disjoint paths. A picture is given below



Observe that any two of the paths form a cycle in the graph. If two of the paths have odd length, then together they form a cycle of length odd+odd = even, as required. So at most one of the paths has odd length. But then the two other paths have even length, and form a cycle of length even+even = even, as required. So in all cases, we get the even cycle.

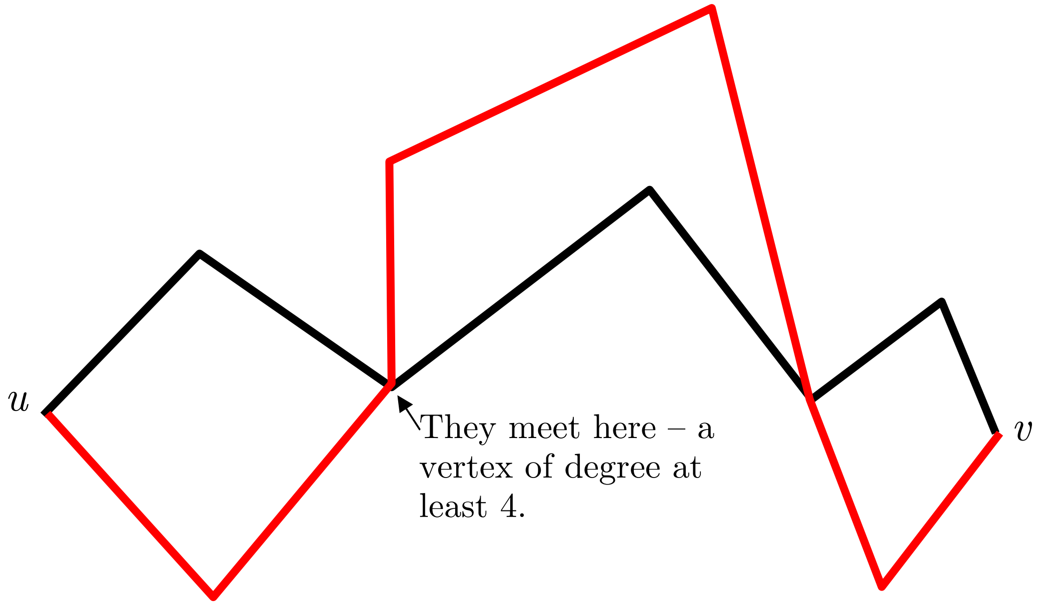
**Example 3.** Show that if a graph has maximum degree at most 3, then $\kappa(G) = \lambda(G)$.

**Solution.**

Use Menger's Theorem. Remember

$$\begin{aligned}\kappa(G) &= \min \kappa(u, v)\\ \lambda(G) &= \min \lambda(u, v)\end{aligned}$$

Now $\lambda(u, v)$, by Menger's Theorem, is the maximum number of edge-disjoint $uv$-paths. If two edge-disjoint paths meet at an internal vertex i.e. a vertex other than $u$ or $v$, then that vertex would have degree at least four (two edges from each path – see below). This contradicts that the maximum degree is at most three. Therefore no two edge-disjoint paths meet internally, which means they are actually internally disjoint. Therefore $\kappa(u, v) = \lambda(u, v)$ for all $u$, $v$, which means $\kappa(G) = \lambda(G)$.



**Example 4*** Show that if all vertices of $G$ have even degree, then $\lambda(G)$ must be even, but $\kappa(G)$ can be odd.

**Solution.**

$\kappa(G)$ can be odd as seen by pasting together two graphs with all vertices of even degree at a vertex. Alternatively, look at the tree of triangles in example 1 where $\kappa(G) = 1$ but all vertices have even degree.

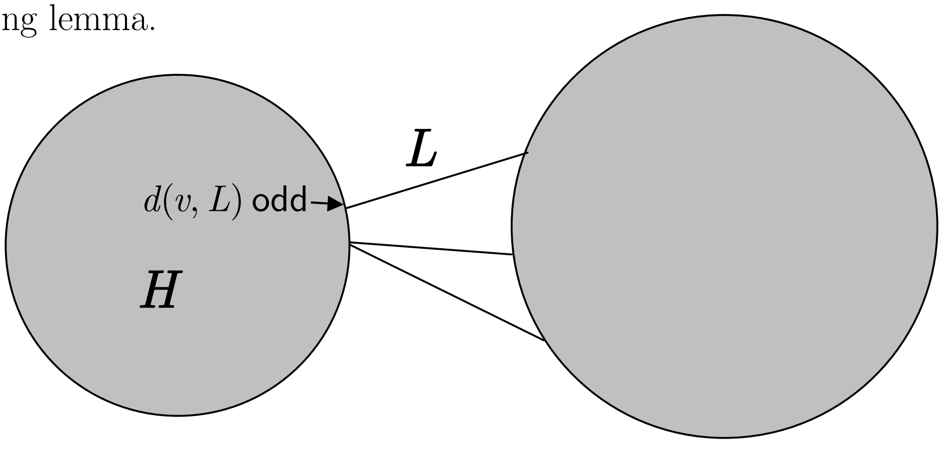
Now we see why $\lambda(G)$ is even. Let $L$ be a minimum edge cut, so

$$| L | = \lambda(G).$$

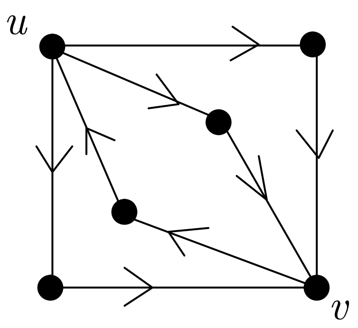
Let $H$ be a component of $G - L$. For each vertex $v$ of $H$, let $d(v, L)$ be the number of edges on $v$ which are in $L$. Then since $L$ is a minimum edge cut

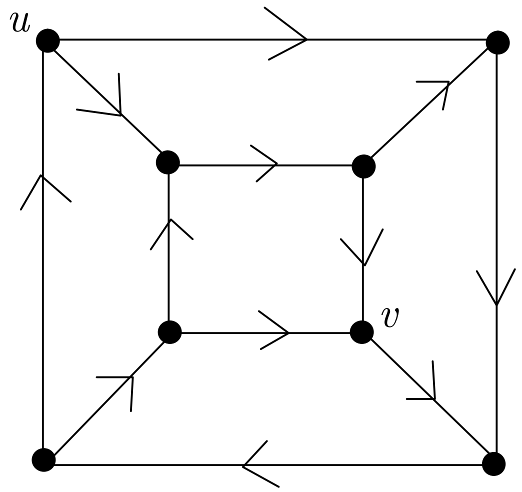
$$\sum_{v \in V(H)} d(v, L) = \left|L\right|.$$

Since this quantity is odd, that means that an odd number of $d(v, L)$ degrees must be odd. But those vertices then have odd degree in H, So H contains an odd number of vertices of odd degree, contradicting the handshaking lemma.



**Example 5.** For a digraph $\vec{G} = (V, \vec{E})$, $u, v \in V$, define $\lambda(u, v)$ to be the minimum number of arcs that must be removed so that there is no directed path from $u$ to $v$. Let $\lambda(\vec{G}) = \min \lambda(u, v)$. Determine $\lambda(u, v)$ and $\lambda(\vec{G})$ for the digraphs below.





For the first digraph, $\lambda(u, v) \le 3$ since we can remove all the arcs which go out of $u$ and then there will be no directed path from $u$ to $v$. On the other hand, by Menger's Theorem for digraphs (Arc Form), $\lambda(u, v)$ is the maximum number of arc-disjoint directed $uv$-paths. We can easily find three such paths using the three arcs out of $u$, so $\lambda(u, v) \ge 3$. So $\lambda(u, v) = 3$.

$\lambda(\vec{G}) = 1$ since we can remove the outgoing arc from any of the vertices between $u$ and $v$, and then that vertex has no directed path out of it.

For the second digraph, $\lambda(u, v) \le 2$ since we can remove the two arcs out of $u$ to destroy all directed $uv$-paths. On the other hand, we can find two arc-disjoint directed $uv$-paths so $\lambda(u, v) = 2$. Also $\lambda(\vec{G}) = 1$ since many vertices have outdegree 1 and we can remove an outgoing arc.