A note on polynomials
and
f-factors of graphs

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Abstract

Let $G = (V,E)$ be a graph, and let $f : V \to 2^{\mathbb{Z}}$ be a function assigning to each $v \in V$ a set of integers in $\{0,1,2,\dots,d(v)\}$, where $d(v)$ denotes the degree of v in G. Lovász [5] defines an f-factor of G to be a spanning subgraph H of G in which $d_H(v) \in f(v)$ for all $v \in V$. Using the combinatorial nullstellensatz [2], we prove that if $|f(v)| > \lceil \frac{1}{2}d(v) \rceil$ for all $v \in V$, then G has an f-factor. This result is best possible and verifies a conjecture of Addario-Berry, Dalal, Reed and Thomason [1].

1 Introduction

In this paper, we are interested in the existence of f-factors of graphs, where the term "f-factor" has a more general meaning than the traditional one. Let $G = (V,E)$ be a graph, and let f be a function assigning to each $v \in V$ a set $f(v)$ of integers. Lovász [5] defines an *f-factor* of G (sometimes called a generalized f-factor) to be a subgraph of G in which $d(v) \in f(v)$ for all $v \in V$. In the case $|f(v)| = 1$, Tutte's f-factor theorem gives a necesary and sufficient condition for the existence of an f-factor of G. No necessary and sufficient condition for an f-factor exists when we allow $|f(v)| \geq 2$, even when $|f(v)| = 2$ for all $v \in V$, since the decision problem of determining whether a graph has an f-factor is known to be algorithmically hard in this case. On the other hand, Lovász [5] showed that if no two consecutive integers in $f(v)$ differ by more than two, then there is a necessary and sufficient condition for an f-factor. We prove the following sufficient condition for f-factors, which verifies a conjecture in Addario-Berry, Dalal, Reed and Thomason [1]:

Theorem 1 *Let $G = (V, E)$ be a graph and suppose that f satisfies*

$$|f(v)| > \lceil d(v)/2 \rceil$$

for every $v \in V$. Then G has an f-factor.

Consider a complete bipartite graph $K_{2n,2n}$ where $f(v) = \{0, 1, 2, \dots, n\}$ for all vertices in one part and $f(v) = \{n, n + 1, \dots, 2n\}$ for all vertices in the other, except one vertex v^* which has $f(v^*) = \{n+1, n+2, \dots, 2n\}$. This graph has no f-factor and the condition of Theorem 1 fails only for v^*.

Alon, Friedland and Kalai [3] showed that in any graph $G = (V, E)$ with more than $(p - 1)|V|$ edges, where p is a prime power, we have a subgraph in which all the degrees are positive multiples of p (this subgraph certainly need not be a spanning subgraph). For example, every $2p - 1$-regular graph contains a p-regular subgraph when p is a prime power. More generally, we can ask for a subgraph where the degrees are in prescribed sets. With this in mind, we define a *partial f-factor* of a graph $G = (V, E)$ to be a $\tilde{f}$-factor of G where $\tilde{f}(v) = f(v) \cup \{0\}$ for all $v \in V$, and a partial f-factor of G is *non-trivial* if it is non-empty. Our next result is a sufficient condition for a graph to have a partial f-factor:

Theorem 2 *Let $G = (V, E)$ be a graph, and let f satisfy*

$$|E| > \sum_{v \in V} |f(v)^c \backslash \{0\}|$$

where $f(v)^c = \{0, 1, 2, \dots, d(v)\} \backslash f(v)$. Then G contains a non-trivial partial f-factor.

We remark that Theorem 2 can be extended easily to hypergraphs, in fact the same sufficient condition as displayed in Theorem 2 works to give an f-factor of a hypergraph $G = (V, E)$. The theorem, even in this generality, is best possible. Take, for instance, any 2-colourable hypergraph $G = (V, E)$ in which every vertex has even degree, and for any 2-colouring (A, B) of G, assign $f(a) = \{0, 1, 2, \dots, d(a)/2\}$ and $f(b) = \{d(b)/2+1, \dots, d(b)\}$ for $(a, b) \in A \times B$. Then G clearly has no non-trivial partial f-factors, and

$$\sum_{v \in V} |f(v)^c \backslash \{0\}| = |E|.$$

Theorem 1 can also be extended to hypergraphs, but the appropriate condition on $|f(v)|$ then depends on the upper rank of the hypergraph – that is, the size of the largest hyperedge.

The proofs of Theorems 1 and 2 involve the use of the combinatorial nullstellensatz [2].

2 Proof of Theorems 1 and 2

The *total degree* of a polynomial $g \in \mathbb{F}[X_1, X_2, \dots, X_n]$ is the largest value of $t_1 + t_2 + \dots + t_n$, taken over all monomials of the form $X_1^{t_1} X_2^{t_2} \dots X_n^{t_n}$ with non-zero coefficient in g. We shall make use of the combinatorial nullstellensatz [2] to prove Theorems 1 and 2:

Theorem 3 (Combinatorial Nullstellensatz) *Let $g \in \mathbb{F}[X_1, X_2, \dots, X_n]$ be a polynomial, and suppose the coefficient of the monomial $\prod_{i=1}^n X_i^{t_i}$ in g is non-zero, where $t_1 + t_2 + \dots + t_n$ is the total degree of g. Then, for any sets $S_1, S_2, \dots, S_n \subset \mathbb{F}$ with $|S_1| > t_1$, $|S_2| > t_2, \dots, |S_n| > t_n$, there exists $x \in S_1 \times S_2 \times \dots \times S_n$ such that $g(x) \neq 0$.*

When applying the nullstellensatz, one has to take care in two steps: the first is defining the right polynomial for the problem, and then to prove algebraically or combinatorially that the largest degree monomial has non-zero coefficient.

Proof of Theorem 1. Consider the polynomial

$$g = \prod_{v \in V} \prod_{c \in f(v)^c} \Big(\sum_{e \ni v} X_e - c \Big)$$

where $f(v)^c = \{0, 1, 2, \dots, d(v)\} \backslash f(v)$ and $X_e \in \{0, 1\}$. We claim that a largest degree monomial in g is of the form $a \prod_{e \in E} X_e^{t_e}$ where $t_e \in \{0, 1\}$ for all $e \in E$ and $a > 0$. The degree of g is exactly

$$\sum_{v \in V} |f(v)^c|.$$

To find a monomial of the required form, we observe that every graph on V has an orientation such that the outdegree of every vertex $v \in V$ is at least $\lfloor \frac{1}{2} d(v) \rfloor$. This can be seen by orienting an eulerian tour in the graph G^* obtained from G by adding a new vertex and joining it to all vertices of G of odd degree. Therefore it is possible to assign to each $v \in V$ a set $E(v)$ of edges containing v such that $|E(v)| = |f(v)^c|$ and, for all distinct $u, v \in V$,

$$E(u) \cap E(v) = \emptyset.$$

Then

$$\prod_{v \in V} \prod_{e \in E(v)} X_e$$

is a monomial of the required degree in g. By the combinatorial nullstellensatz, there exists $x \in \{0, 1\}^{|E|}$ such that $g(x) \neq 0$. Now $F = \{e \in E : x_e = 1\}$ is the edge set of an f-factor of G.

Proof of Theorem 2. Consider the polynomial over $\mathbb{R}$ defined by

$$g \;=\; \prod_{v \in V} \prod_{c \in f(v)^c \backslash \{0\}} \left(\frac{c - \sum_{e \ni v} X_e}{c} \right) - \prod_{e \in E} (1 - X_e).$$

and where $X_e \in \{0,1\}$ for all $e \in E$. Then $g(0) = 1 - 1 = 0$. By the inequality of the theorem, the total degree of the first term in g is

$$\sum_{v \in V} |f(v)^c \backslash \{0\}| \;\; < \;\; |E|$$

so we conclude that the largest degree monomial in g is precisely $(-1)^{|E|+1} \prod_{e \in E} X_e$. By the combinatorial nullstellensatz, there exists a non-zero $x \in \{0,1\}^{|E|}$ such that $g(x) \neq 0$. This implies that the first term in g is not zero at x, and so $\sum_{v \ni e} x_e \in f(v) \cup \{0\}$ for all $v \in V$. If

$$F \;\; = \;\; \{e \in E : x_e = 1\},$$

then F is the edge set of a non-trivial partial f-factor of G.

3 Concluding Remarks

• We mention a possible direction for extending the results of Theorem 1 and Theorem 2. Alon, Friedland and Kalai [3] proved that for any prime power p, every graph of average degree more than $2p-2$ contains a non-empty subgraph all of whose vertices have degree zero modulo p. Let $f : V \to \mathbb{Z}$ be a function assigning to each $v \in V$ a set of integers. A natural extension of Theorem 2 is to ask, given a graph $G = (V, E)$ and a function $f : V \to \mathbb{Z}$, for a subgraph $H = (W, F)$ such that $W \neq \emptyset$ and the degree of a vertex w in H is congruent modulo p to $f(w)$, for all $w \in W$ (we might call this a *partial f-factor mod p*). We ask whether such a subgraph is guaranteed to exist in a graph with sufficiently large (but constant) average degree – even the case $f(v) = \{q\}$ for all $v \in V$ appears to be challenging. The question of finding sufficient conditions for a subgraph in which the degree of v is $f(v)$ modulo p for every vertex $v \in V$ (we might call such a subgraph an *f-factor mod p*) appears to be interesting.

• In [1], the problem of vertex-distinguishing edge-colourings is considered. Karoński, Luczak and Thomason [4] conjectured that given any graph $G = (V, E)$ with no component isomorphic to K_2, there is a labelling of the edges with the integers from $\{1, 2, 3\}$ such that the sum of labels of the edges containing $u \in V$ and the sum of labels of the edges containing $v \in V$ are distinct whenever $\{u, v\} \in E$. A polynomial which might be useful for this problem is

$$g = \prod_{\{u,v\} \in E} \Big(\sum_{e \ni u} x_e - \sum_{f \ni v} x_f \Big)^2$$

where $x = (x_e)_{e \in E}$ and $x_e \in \{1, 2, 3\}$ for all $e \in E$. Note that if $g(x) \neq 0$, then the entries of x give the required labelling of E. We ask whether the coefficient of $\prod_{e \in E} x_e^2$ is non-zero in g. By the nullstellensatz, this guarantees that $g(x) \neq 0$ for some x.

References

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