

# Note - Binomial Coefficients

jacques@ucsd.edu

It is important to be familiar with manipulating binomial coefficients, so we give some basic facts and identities governing them. You should be able to prove all the identities given

**Fact 1.** The number of subsets of an  $n$ -element set of size  $k$  is defined to be

$$\binom{n}{k}$$

**Fact 2.** A combinatorial proof shows

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

**Fact 3.** Arithmetic on Fact 2 gives

$$\binom{n}{k} = \frac{n(n-1)(n-2)\dots(n-k+1)}{k!}$$

**Fact 4.** From Fact 1 we get

$$\binom{n}{0} = 1 = \binom{n}{n} \quad \text{and} \quad \binom{n}{k} = 0 \text{ if } k > n$$

**Fact 5.** Symmetry of binomial coefficients

$$\binom{n}{k} = \binom{n}{n-k}$$

**Fact 6.** The generalized binomial coefficient for  $a \in \mathbb{Q}$  and  $k \in \{0, 1, 2, \dots\}$  is defined by

$$\binom{a}{k} = \frac{a(a-1)(a-2)\dots(a-k+1)}{k!}$$

**Fact 7.**

$$\binom{-1}{k} = (-1)^k \quad \text{and} \quad \binom{-n}{k} = (-1)^k \binom{n+k-1}{k}.$$

**Fact 8.** Binomial Theorem: for  $a \in \mathbb{Q}$ :

$$(1+x)^a = \sum_{k=0}^{\infty} \binom{a}{k} x^k$$

**Fact 9.** Basic identities

[Pascal's Triangle]	$\binom{n}{k}$	$=$	$\binom{n-1}{k} + \binom{n-1}{k-1}$
[Total number of subsets]	$\sum_{k=0}^n \binom{n}{k}$	$=$	$2^n$
[Number of subsets of odd and even size are equal]	$\sum_{k \text{ odd}} \binom{n}{k}$	$=$	$\sum_{k \text{ even}} \binom{n}{k}$