Clique Partitions of Dense Graphs

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Abstract

In this paper, we prove that for any forest \( F \subset K_n \), the edges of \( E(K_n) \setminus E(F) \) can be partitioned into \( O(n \log n) \) cliques. This extends earlier results on clique partitions of the complement of a perfect matching and of a hamiltonian path in \( K_n \). We also show that if a graph \( G \) has maximum degree \( \Delta \), then the edges of \( E(K_n) \setminus E(G) \) can be partitioned into roughly \( n^{3/2} \Delta^{1/2} \log^2 n \) cliques provided there exist Steiner systems with certain parameters. Furthermore, as \( n \to \infty \), almost every \( \Delta \)-regular graph \( G \) on \( n \) vertices has the property that \( E(K_n) \setminus E(G) \) cannot be partitioned into fewer than about \( \Delta^2 \frac{\log \Delta}{\log \log \Delta} - 1 \) cliques.

1 Introduction

A clique partition of a graph \( G \) is a collection of complete subgraphs of \( G \) (called cliques) that partition the edge set of \( G \). In this paper, we study the problem of finding clique partitions of \( K_n \setminus F \), where \( F \subset K_n \) is a forest or a graph of maximum degree \( \Delta \). Here \( K_n \setminus F \) refers to the graph on \( V(K_n) \) consisting of all edges of \( K_n \) which are not in \( F \), and is called the complement of \( F \). We denote by \( \text{cp}(K_n \setminus F) \) the clique partition number of \( K_n \setminus F \), which is the smallest number of cliques partitioning \( E(K_n) \setminus E(F) \). Any further notation not defined here is found in Bondy and Murty [6]. Gregory, McGuinness and Wallis [7] proved that the complement of a perfect matching on \( n \) vertices can be partitioned into \( O(n \log \log n) \) cliques. In the case that \( F \) is a forest, we prove the following theorem:

**Theorem 1** Let \( F \subset K_n \) be a forest. Then \( \text{cp}(K_n \setminus F) = O(n \log n) \).
The proof of Theorem 1 will be given in Section 2. We are not aware of any lower bounds for \( \text{cp}(K_n \setminus F) \) which are not of order of magnitude \( O(n) \); we conjecture that there exist forests for which
\[
\text{cp}(K_n \setminus F) \rightarrow \infty.
\]

In the second part of the paper, we are interested in estimating clique partition numbers of very dense graphs – we wish to find bounds on \( \text{cp}(K_n \setminus G) \) when \( G \) has maximum degree \( \Delta \). An early result of Erdős and de Bruijn [4] shows that \( \text{cp}(K_n \setminus K_2) = n - 1 \) for \( n \geq 3 \), with equality only for clique partitions consisting of \( n - 2 \) complete graphs of order two and one complete graph of order \( n - 2 \). Using projective planes, Wallis [8] showed if \( H \) is a graph with at most \( \sqrt{n} \) vertices, then \( K_n \setminus H \) can be partitioned into \( O(n) \) cliques. A projective plane is a particular example of a family of sets called a Steiner system. We recall that a Steiner \((n, k)\)-system provides a clique partition of \( K_n \) into cliques of size \( k \); in particular \( \binom{n}{2}/\binom{k}{2} \) cliques are present in this partition. Conditional on the existence of Steiner systems with certain parameters, we give bounds on \( \text{cp}(K_n \setminus G) \) for graphs \( G \) with prescribed maximum degree \( \Delta(G) \):

**Theorem 2** Let \( G \) be a graph on \( n \) vertices, let \( k = \lfloor (\frac{n}{\Delta})^{1/2} \rfloor \), and suppose there exists a Steiner \((n, k)\)-system. Then, provided that \( \Delta(G) = o(\frac{n}{\log n}) \) as \( n \) tends to infinity,
\[
\text{cp}(K_n \setminus G) = O(n^{1.5} \Delta(G)^{1.5} \log^2 n).
\]
Furthermore, if \( \frac{n - \Delta}{\log n} \rightarrow \infty \), then as \( n \rightarrow \infty \), almost every \( \Delta \)-regular graph \( G \) on \( n \) vertices has
\[
\text{cp}(K_n \setminus G) = \Omega(\Delta^{2\lfloor \frac{\log n}{\log 2\Delta} \rfloor - 1}).
\]

For the purpose of comparison, if \( \Delta = n^{1-\epsilon} \) in Theorem 2, then the upper bound for \( \text{cp}(K_n \setminus G) \) is of order \( n^{2-\frac{1}{2}\epsilon} \log^2 n \) whereas the lower bound is of order \( n^{2-2\epsilon} \). It would be interesting to determine whether either of these bounds is tight in order of magnitude. We conclude with the following conjecture:

**Conjecture 3** Let \( G \) be a graph on \( n \) vertices with maximum degree \( \Delta \). Then \( \text{cp}(K_n \setminus G) = O(\Delta n \log n) \). Furthermore, if \( \Delta = o(n) \), then \( \text{cp}(K_n \setminus G) = o(n^2) \).

## 2 Complements of Forests

To prove Theorem 1, we will restrict our attention to trees and show \( \text{cp}(K_n \setminus T) = O(n \log n) \) for any tree \( T \) on \( n \) vertices. Theorem 1 follows from this statement, since every forest \( F \subset K_n \) is contained in a spanning tree \( T \subset K_n \), and
\[
\text{cp}(K_n \setminus F) \leq \text{cp}(K_n \setminus T) + n - 1.
\]
To prove the claim \( \text{cp}(K_n \setminus T) = O(n \log n) \), we make use of the following definition.
Let of every prime order, we can take results on the distribution of prime numbers \[2\], and the fact that there exists a projective plane smooth tree partition starting with the trivial tree partition, repeatedly take 2-smooth partitions of all trees of size more than done. Without loss of generality, suppose that for \(i = 1, 2, \ldots, r\). The following lemma, which is easily seen to be best possible, will be used to prove Theorem 1.

**Definition 4** A tree partition of a graph \(G\) is a collection of subtrees \(\{T_1, T_2, \ldots, T_r\}\) of \(G\) such that every edge of \(G\) is in exactly one subtree:

\[
G = \bigcup_{i=1}^{r} T_i,
\]

and for all \(i \neq j\), \(|V(T_i) \cap V(T_j)| \leq 1\).

For a positive integer \(b\), we say a tree partition \(\{T_1, T_2, \ldots, T_r\}\) is \(b\)-smooth if for some \(k\), \(k \leq b|T_i| \leq bk\), for \(i = 1, 2, \ldots, r\). The following lemma, which is easily seen to be best possible, will be used to prove Theorem 1.

**Lemma 5** Let \(T\) be a tree on \(n\) vertices and let \(2 \leq k \leq n\). Then there exists a 3-smooth tree partition of \(T\) into at most \(2n/k\) trees such that every tree in the partition has size at most \(k\).

**Proof.** Let \(T\) be a tree on \(n\) vertices. It is well known that there exists a 2-smooth tree-partition \(\{T_1, T_2\}\) of \(T\). To see this, take a tree partition \(\{T_1, T_2\}\) of \(T\) so that \(|V(T_1)| - |V(T_2)|\) is minimized, and assume \(T_1\) and \(T_2\) share vertex \(v\). If \(\frac{2}{3} \leq |T_i| \leq \frac{2n}{3}\), for \(i = 1, 2\), then we are done. Without loss of generality, suppose that \(|T_1| < \frac{2n}{3}\). As \(|V(T_1)| - |V(T_2)|\) is minimized, \(v\) is adjacent to at least two vertices of \(T_2\). Form a tree partition \(\{J_1, J_2\}\) of \(T_2\), such that \(J_1\) and \(J_2\) share vertex \(v\). Then

\[
\frac{2n}{3} + 1 < |T_1 \cup J_1| + |T_1 \cup J_2| < \frac{4n}{3} + 1.
\]

Then \(\{T_1 \cup J_1, J_2\}\) or \(\{T_1 \cup J_2, J_1\}\) is a 2-smooth tree partition of \(T\).

To finish the proof, we construct a 3-smooth tree partition of \(T\) into trees of size at most \(k\). Repeatedly take 2-smooth partitions of all trees of size more than \(k\) in the current tree-partition, starting with the trivial tree partition, \(\{T\}\). This procedure gives a tree partition of \(T\) all of whose trees have size at most \(k\) and at least \(k/3\), and average size at least \(k/2\) as required.

We now use Lemma 5 to prove Theorem 1.

**Proof of Theorem 1.** Define for \(n \in \mathbb{N}\):

\[
g(n) = \max\{\text{cp}(K_n \setminus T) : T \text{ is a tree on } n \text{ vertices}\}.
\]

Let \(T\) be a tree on \(n\) vertices such that \(g(n) = \text{cp}(K_n \setminus T)\). By Lemma 5, there exists a 3-smooth tree partition \(\{T_1, T_2, \ldots, T_r\}\) of \(T\) such that \(|T_i| \leq \sqrt{n}\) and \(r \leq 2\sqrt{n}\). Without loss of generality, suppose that for \(i = 2, 3, \ldots, r\),

\[
V(T_i) \cap \left( \bigcup_{j=1}^{i-1} V(T_j) \right) = \{v_i\}.
\]

Let \(t\) be the smallest integer such that \(t \geq 7\sqrt{n}\) and there is a projective plane of order \(t\). By results on the distribution of prime numbers \([2]\), and the fact that there exists a projective plane of every prime order, we can take \(t = 7\sqrt{n} + n^\theta\) for some \(\theta < \frac{1}{2}\) and choose a projective plane of
order \( t \). We identify the \( t^2 + t + 1 \) points of the projective plane with the vertices of a complete graph \( K_{t^2+t+1} \), and the blocks form a clique partition of this complete graph. We claim that we can embed the trees \( T_1, T_2, \ldots, T_r \) in the cliques \( B_1, B_2, \ldots, B_r \) in such a way that the union of these embedded trees is \( T \).

First identify the vertices of \( T_1 \) with points from an arbitrary block, say \( B_1 \), of the projective plane, where the vertex \( v_2 \) is identified with some point \( w_2 \), and all other vertices of \( T_1 \) are identified arbitrarily with points from \( B_1 \) \( \setminus \{w_2\} \). Suppose that for some \( i : 2 \leq i \leq r \), we have already identified the vertices of \( T_{i-1} \) with the points of \( B_{i-1} \), such that vertex \( v_i \) is identified with point \( w_i \) of some block \( B_j \), where \( j \leq i - 1 \). Pick a block \( B_i \) (different from \( B_1, B_2, \ldots, B_{i-1} \)) that contains the point \( w_i \). There exists such a block as there are \( t + 1 \geq 7\sqrt{n} \) blocks containing the point \( w_i \) (and at most \( r \leq 2\sqrt{n} \) blocks have been used). Identify the vertices of \( T_i \) with points from \( B_i \) such that \( v_i \) is identified with \( w_i \), and all other vertices of \( T_i \) are identified arbitrarily with points from \( B_i \setminus W \), where \( W \) is the set of points from \( B_1 \cup B_2 \cup \cdots \cup B_{i-1} \) that intersect with \( B_i \). Note that \( |W| < r \), as \( B_i \) intersects every other block in at most one point. This identification can be done, as each block has \( t + 1 \geq 7\sqrt{n} \) points, each tree has at most \( \lfloor \sqrt{n} \rfloor \) vertices, and removing at most \( r - 1 \leq 2\sqrt{n} \) points of block \( B_i \) leaves at least \( \sqrt{n} \) points which can be identified with \( T_i \). This defines the embedding of the tree \( T_i \) into \( B_i \) for \( i = 1, 2, \ldots, r \), and the union of the embedded trees is clearly \( T \).

Now delete points of the projective plane such that each block \( B_i \) has \( \lfloor \sqrt{n} \rfloor \) points. Then:

\[
g(n) \leq t^2 + t + 1 - r + \sum_{i=1}^{r} \text{cp}(K_{|B_i|} \setminus T_i)
\]

\[
= O(n) + \sum_{i=1}^{r} \text{cp}(K_{\lfloor \sqrt{n} \rfloor} \setminus T_i)
\]

\[
\leq O(n) + \sum_{i=1}^{r} g(\lfloor \sqrt{n} \rfloor)
\]

Defining \( c(x) = g(\lfloor x \rfloor) \) for \( x \in \mathbb{R} \) gives

\[
c(x) \leq O(x) + 2\sqrt{x} \cdot c(\sqrt{x}).
\]

Dividing through by \( x \) and setting \( z = \log_2 \log_2 x \), and \( h(z) = c(x)/x \) gives,

\[
h(z) \leq O(1) + 2 \cdot h(z - 1).
\]

So, \( h(z) = O(2^z) \), for \( x \) (and hence \( z \)) arbitrarily large. Hence, \( c(x) = O(x \log x) \) implying that \( g(n) = O(n \log n) \).

3 Dense Graphs

In this section, we prove Theorem 2 using Steiner systems (see Cameron and van Lint [5]) and the probabilistic method (see Alon and Spencer [1]).
Recall that the blocks of a Steiner $(n, k)$-system correspond to a clique partition of $K_n$ into $\binom{n}{2}/\binom{k}{2}$ cliques of size $k$. Necessary existence conditions for the existence of an $S(n, k)$ are

$$n \equiv 1 \mod k - 1,$$

$$n(n - 1) \equiv 0 \mod k(k - 1).$$

Wilson’s theorem [3] says that the necessary conditions above for the existence of an $S(n, k)$ are sufficient for almost all $n \in \mathbb{N}$. However, the proofs presented by Wilson do not give an explicit constant $n_0(k)$ such that an $S(n, k)$ exists for all $n \geq n_0(k)$ satisfying the necessary conditions. Recently, Chang showed that $n_0(k) \leq \exp(\exp(k^2))$ (see page 800 in Beth, Jungnickel and Lenz [3]). It is therefore out of the reach of current research to determine for which $k \in \{1, 2, \ldots, n\}$ a Steiner $(n, k)$-system exists.

**Proof of Theorem 2.** Both the proofs of the upper and lower bounds in Theorem 2 are probabilistic. First we show that for almost every $\Delta$-regular graph $G$ on $n$ vertices,

$$\text{cp}(K_n \setminus G) = \Omega(\Delta^2 [\log n/\log \Delta]^2 - 1^2).$$

Note that this statement is true whenever the right hand side is of order $n$, since $\text{cp}(K_n \setminus G) = \Omega(n)$ for every $\Delta$-regular graph $G$. For this part of the proof, let $m = m(\Delta, n)$ be an integer such that as $n \to \infty$,

$$m = o(\Delta^2 [\log n/\log \Delta]^2 - 1^2).$$

Note that $\Delta^2 [\log n/\log \Delta]^2 - 1^2 \to \infty$ since $\frac{n - \Delta}{\log n} \to \infty$. For each clique partition $C$ of a graph $G$ on $n$ vertices, let $B(G, C)$ be the bipartite graph with parts $C$ and $V(G)$ such that $v \in V(G)$ is joined to $C \in C$ if $v \in C$. We observe that $B(G, C)$ has no cycles of length four and all vertices $C \in C$ have degree at least two. Therefore

$$\sum_{C \in \mathcal{C}} \left( \frac{d(C)}{2} \right) \leq \binom{n}{2}.$$

This implies that if $|C| = m$, then the number of edges in $B(G, C)$ is at most

$$\frac{m}{2} + m \left( \frac{1}{4} + \frac{n^2}{m} \right)^{\frac{1}{2}} < 2\sqrt{mn}$$

provided $n$ is large enough.

Let $\beta(m, n, \Delta)$ be the total number of graphs $B(G, C)$ with $|C| = m$. Then

$$\log \beta(m, n, \Delta) \leq \log \sum_{k \leq 2\sqrt{mn}} \binom{mn}{k} = O(\sqrt{mn} \log m)$$

$$= o(\Delta n \log \Delta \cdot [\log n/\log \Delta - 1])$$

by definition of $m$. By the results on random regular graphs in Wormald [9], the logarithm of the number $\gamma(n, \Delta)$ of $\Delta$-regular graphs on $n$ vertices is at least

$$\log \gamma(n, \Delta) \geq \log \left( \frac{(\Delta n)!}{2^\Delta \Delta^n (\frac{\Delta n}{2})! (\Delta^n)^n} \cdot \exp[-\Omega(\Delta^2)] \right) = \Omega(\Delta n \log \frac{n}{\Delta})$$

$$= \Omega(\Delta n \log \Delta \cdot [\log n/\log \Delta - 1]).$$
We conclude that for any $m$ as defined above, and since $n - \frac{\Delta}{\log n} \to \infty$,

$$\lim_{n \to \infty} \frac{\log \beta(m, n, \Delta)}{\log \gamma(n, \Delta)} = 0.$$ 

In particular, the proportion of $\Delta$-regular graphs $G \subset K_n$ such that $\text{cp}(K_n \setminus G) = m$ tends to zero exponentially fast in $n$ as $n \to \infty$. It follows that almost all $\Delta$-regular graphs $G$ on $n$ vertices have

$$\text{cp}(K_n \setminus G) = \Omega(\Delta^2 \left[ \frac{\log n}{\log \Delta} - 1 \right]^2).$$

This completes the first part of the proof.

Now we prove the upper bound on $\text{cp}(K_n \setminus G)$ given in Theorem 2. Suppose $G, n, k$ satisfy the conditions of the theorem. Let $\mathcal{S} = (X, B)$ be a Steiner system with blocks of size $k$ on $n$ points. For a random permutation of the points, the probability that a fixed set of $k$ points is a fixed block in $\mathcal{B}$ is exactly $1/(\binom{n}{k})$. Take $G$ to be a fixed graph on the same set of $n$ points, with maximum degree $\Delta$. Let $G_B$ denote the subgraph of $G$ spanned by the edges contained in a block $B \in \mathcal{B}$. Consider the event $|E(G_B)| \geq r$, for some integer $r$. Pick a subgraph $H_B$ of $G_B$ with exactly $r$ edges. If the maximum size of a matching in $H_B$ is $i$ for some positive integer $i \leq r$, and if there are $s$ vertices of $H_B$ which are unsaturated by a maximum matching, then

$$\max \left\{ \frac{r - \left(\frac{2i}{2}\right)}{2i}, 0 \right\} \leq s \leq r - i.$$

For convenience, let $s_i = (r - \left(\frac{2i}{2}\right))/2i$. Let $A_B(i, s)$ denote the event that the largest matching in $H_B$ has size $i$ and $H_B$ has $s + 2i$ vertices. Fixing a matching $M$ of size $i$ in $G$, there are at most $(2i)^s$ ways to choose a set $S$ of $s$ vertices so that $V(M) \cup S = H_B$. Then there are $(\begin{array}{c} n - 2i - s \\ k - 2i - s \end{array})$ ways to choose the vertices of $B \setminus V(H_B)$ from $G$. Therefore

$$\mathbb{P}[A_B(i, s)] \leq \frac{1}{\binom{n}{k}} \frac{\Delta^n}{i} (2i)^s \binom{n - 2i - s}{k - 2i - s}.$$

Since $\{|E(G_B)| \geq r\} \subset \bigcup_{i, s} A_B(i, s)$, it follows that

$$\mathbb{P}[|E(G_B)| \geq r] \leq \frac{1}{\binom{n}{k}} \sum_{i=1}^{r} \sum_{s \geq s_i} (\Delta^n)_i (2i)^s \binom{n - 2i - s}{k - 2i - s}.$$

To estimate the sums on the right, we use the inequality

$$\frac{\binom{n-i}{k-i}}{\binom{n}{k}} \leq \frac{b^i}{a^i}.$$

Let $j$ be the largest integer such that

$$r - \left(\frac{2j}{2}\right) \geq 0$$
so that definitely $\sqrt{r/2} - 1 \leq j \leq \sqrt{r/2} + 1$. Recall $k = \lceil \sqrt{n/2\Delta} \rceil$. Then

$$
P[|E(G_B)| \geq r] \leq \sum_{i=1}^r \sum_{s \geq s_i, s \geq 0} \binom{n}{i} (2i\Delta)^s \binom{n-2i-s}{k-2i-s} \binom{n}{k}
$$

$$
< \sum_{i=1}^r \sum_{s \geq s_i, s \geq 0} (\Delta n)^i (2r\Delta)^s \frac{k^{2i+s}}{n^{2i+s}}
$$

$$
= \sum_{i=1}^r \binom{\Delta n^2}{n} \sum_{s \geq s_i, s \geq 0} \left( \frac{2rk\Delta}{n} \right)^s
$$

$$
\leq \sum_{i=1}^j \left( \frac{1}{2} \right)^i \sum_{s=s_i}^{r-i} \left( \frac{1}{2} \right)^s + \sum_{i=j+1}^r \left( \frac{1}{2} \right)^i \sum_{s=0}^{r-i} \left( \frac{1}{2} \right)^s
$$

$$
\leq 2 \sum_{i=1}^j \left( \frac{1}{2} \right)^i \left( \frac{1}{2} \right)^{s_i} + 4 \left( \frac{1}{2} \right)^{j+1}
$$

$$
= \sqrt{2} \sum_{i=1}^j \left( \frac{1}{2} \right)^{r/2i} + \left( \frac{1}{2} \right)^{j-1}
$$

$$
\leq \sqrt{2} j \left( \frac{1}{2} \right)^{Z/2j} + \left( \frac{1}{2} \right)^{j-1}.
$$

To justify (*), note that $\frac{2rk\Delta}{n} \leq \frac{1}{2}$ is true if we set $r = 2(\log_2(2n^2\Delta))^2$ and recall that

$$
\frac{\Delta (\log_2 n)^4}{n} \to 0.
$$

In addition, $\frac{\Delta n^2}{n} \leq \frac{1}{2}$ by our choice of $k$. These two statements verify (*). We show that the above expression is less than $\frac{1}{|B|}$. That is, we prove

$$
\left[ \sqrt{2} j \left( \frac{1}{2} \right)^Z + \left( \frac{1}{2} \right)^{j-1} \right] \frac{n(n-1)}{k(k-1)} < 1.
$$

But, the left hand side of this is

$$
\left[ \sqrt{2} j \left( \frac{1}{2} \right)^Z + \left( \frac{1}{2} \right)^{j-1} \right] \frac{n(n-1)}{k(k-1)} = O \left( \frac{jn^2}{k^2} \left( \frac{1}{2} \right)^Z \right)
$$

$$
= O \left( \Delta n \log_2(2n^2\Delta) \left( \frac{1}{2} \right)^{2\log_2(2n^2\Delta)+1} \right)
$$

$$
= O \left( \Delta n \log_2(2n^2\Delta) \left( \frac{1}{2} \right)^{\log_2(2n^2\Delta)} \right)
$$

$$
= O \left( \frac{\log_2(2n^2\Delta)}{n} \right)
$$

$$
= O(\log_2 n/n).
$$
So with positive probability $|E(G_B)| < r$ for all $B \in \mathcal{B}$, and
\[
\text{cp}(K_n \setminus G) < r|\mathcal{B}| = O(n^{3/2} \Delta^{1/2} \log^2 n),
\]
if $n$ is sufficiently large. This completes the proof.

References


