Basic Principles of Counting

1 The basic objects

The main types of objects we count are sets and sequences. In class, we showed that there are \( n^k \) sequences of length \( k \) with entries from \([n] := \{1, 2, \ldots, n\}\). We use round brackets to denote that the entries of a sequence are ordered. Recall that a permutation is a sequence where all the entries are different. We prove that there are \( n(n-1) \ldots (n-k+1) \) permutations.

In the lectures, we showed that there are \( 2^n \) subsets of \([n]\) and \( \binom{n}{k} \) subsets of \([n]\). We use braces to denote that the elements of a set are unordered. This section is aimed at how to approach problems involving counting sequences and counting sets, by presenting two general principles and examples of how to apply them.

2 Multiplication principle

The first extremely useful step-by-step counting principle is for counting sequences:

**The Multiplication Principle.** The number of sequences \((x_1, x_2, \ldots, x_k)\) is the product of the number of choices of each \(x_i\) for \(i \in [k]\).

For example, how do we see that there are \( n! \) permutations of \([n]\)? Well we do it step-by-step: there are \( n \) choices for the first entry of a permutation, \( n-1 \) choices for the next entry, and so on. By the multiplication principle, we just multiply these numbers together, so there are \( n(n-1)(n-2) \ldots 2 \cdot 1 = n! \) permutations. Another example is to count the number of sequences \((x_1, x_2, \ldots, x_k)\) where \(i \leq x_i \leq 2i\) for all \(i \in [k]\). Clearly there are \( i+1 \) choices for \(x_i\), so the multiplication principle gives us the answer, which is \((k+1)!\). It is extremely important to realize that the multiplication principle does not work for counting sets.

3 Unordering principle

Okay, but then how would we count sets? This leads into our second principle: if we count \( N \) sequences \((x_1, x_2, \ldots, x_k)\) and we know that none of the \(x_i\)s are the same, then the number of sets \(\{x_1, x_2, \ldots, x_k\}\) is just \(\frac{N}{k!}\).

**The Unordering Principle.** If there are \( N \) sequences \((x_1, x_2, \ldots, x_k)\) where none of the \(x_i\)s are equal, then there are \(\frac{N}{k!}\) sets \(\{x_1, x_2, \ldots, x_k\}\).

In other words, we divide by \( k! \) to “get rid of the order”. A good tip, therefore, when counting sets is to first count sequences and then divide by the appropriate factorial using the unordering principle. We stress that none of the \(x_i\)s can be equal when we apply the unordering principle. The fundamental example in applying the unordering principle is to count sets of size \( k \) in \([n] \) – we proved in class that the answer is \(\binom{n}{k}\). Let’s do it: first we count permutations \((x_1, x_2, \ldots, x_k)\) of length \( k \) with entries from \([n]\); by the multiplication
principle there are \( n(n-1)(n-2)\ldots(n-k+1) \) such permutations (check this). By the unordering principle, to get the number of sets of size \( k \) in \([n]\), we just divide the number of permutations of length \( k \) by \( k! \) to get
\[
\frac{n(n-1)(n-2)\ldots(n-k+1)}{k!} = \binom{n}{k}.
\]

4 Partitions

A partition of \([n]\) is a set \( \{A_1, A_2, \ldots, A_k\} \) of non-empty sets whose union is \([n] \) such that no two of the sets intersect. The sets \( A_i \) are called the parts of the partition. It is convenient to write 23 instead of the set \( \{2, 3\} \), otherwise the notation gets a bit ugly. So for example, \( \{1, 23\} \) is a partition of \([3]\). An ordered partition of \([n]\) is an ordering of the sets in a partition of \([n]\). For example, we count \((1, 23)\) and \((23, 1)\) as different ordered partitions of \([3]\), even though \( \{1, 23\} \) is the same (unordered) partition as \( \{23, 1\} \). How many ordered partitions of \([n]\) have their \( i \)th part of size \( n_i \) for \( i \in [k] \)? From the theorem in class, the answer is the multinomial coefficient
\[
\binom{n}{n_1 n_2 \ldots n_k} := \frac{n!}{n_1! n_2! \ldots n_k!} = \frac{n}{n_1} \left(\frac{n-n_1}{n_2}\right) \ldots \left(\frac{n-n_1-n_2-\ldots-n_{k-1}}{n_k}\right).
\]
This follows from the multiplication principle, since the product on the right represents the number of choices for the first part of the partition, then the second part, the third, and so on. But what about unordered partitions of \([n]\) into parts of sizes \( n_1, n_2, \ldots, n_k \)? This problem is hard in general – in general we can’t just divide the number of ordered partitions by \( k! \) – so we do not discuss it here. In the particular case \( n_1 = n_2 = \ldots = n_k = n/k \), we can apply the unordering principle to get the answer
\[
\frac{1}{k!} \left(\frac{n}{m \ m \ \ldots \ m}\right) = \frac{n!}{k!(m!)^k}
\]
since any ordering of the parts of the partition gives an ordered partition with all parts of size \( m \). In the case where all the parts have different sizes – all \( n_is \) are different – the number of unordered partitions is the same as the number of ordered partitions.

5 Compositions

A composition of an integer \( n \) with \( k \) parts is a sequence \((x_1, x_2, \ldots, x_k)\) of positive integers such that \( x_1 + x_2 + \ldots + x_k = n \). The \( x_is \) are called the parts of the composition. To count these compositions, it seems at first we should just apply the multiplication principle: count the number of choices for \( x_1 \), and then the number of choices for \( x_2 \), and so on, and then multiply all these together. But there’s a catch: once we’ve picked \( x_1 \), the number of choices of \( x_2 \) depends on what \( x_1 \) was picked! We have to find a different strategy. This is the third and perhaps trickiest principle: reduce the problem to a different counting problem to which we know the answer. In class, we reduced this problem to counting sets of size \( k-1 \) in \([n-1]\), to which the answer is \( \binom{n-1}{k-1} \). We did that by noticing that if we cut \([n]\) into \( k \) intervals, then the lengths of the intervals are exactly the \( x_is \). So how many ways can we cut \([n]\) into \( k \) intervals? Well we have to choose \( k-1 \) places to cut \([n]\) and there are \( n-1 \) possible
places between numbers where cuts can be made. This validates our answer. We haven’t said anything about counting sets \( \{x_1, x_2, \ldots, x_k\} \) such that \( x_1 + x_2 + \ldots + x_k = n \). This is with good reason: we can’t apply the unordering principle, so we get into the same problems as when we try to count unordered partitions. It turns out that the total number of sets of positive integers whose sum is \( n \) is asymptotic to the strange formula

\[
\frac{1}{4n\sqrt{3}} \cdot e^{\pi\sqrt{\frac{n}{3}}}
\]

We definitely will not go near this formula here!

6 Counting by Correspondence

If we want to prove that two sets \( A \) and \( B \) have the same size, we could pair up (or match) each element of \( A \) to a unique element of \( B \). Formally, a bijection (or one-to-one correspondence) is a function \( f : A \to B \) such that for every \( b \in B \), there is a unique \( a \in A \) such that \( f(a) = b \). The principle here is that if \( f : A \to B \) is a bijection, then \( |A| = |B| \), and sometimes to find \( |A| \) is hard, but finding \( |B| \) is easier. For example, consider counting compositions \((x_1, x_2, \ldots, x_k)\) of \( n \) with \( x_i \geq i \). We let \( A \) be the set of these compositions, and let \( B \) be the set of compositions of \( n - 1 - 2 - \ldots - (k - 1) = n - \binom{k}{2} \). Then the function \( f : A \to B \) defined by

\[
f(x_1, x_2, \ldots, x_k) = (x_1, x_2 - 1, x_3 - 2, \ldots, x_k - (k - 1))
\]

is a bijection (every composition in \( B \) comes from a unique composition in \( A \)), so \( |A| = |B| \). But by the theorem on compositions,

\[
|A| = |B| = \left( n - \binom{k}{2} - 1 \right)
\]

and that is the answer to the problem. Here is another example: suppose we want to count subsets of \( \{1, 2, \ldots, n\} \) of size \( k \) such that no two elements of the subset are consecutive integers. For example, if \( k = 2 \) and \( n = 3 \) then the only possible set is \( \{1, 3\} \). If we arrange the elements \( s_1, s_2, \ldots, s_k \) of such a set in increasing order, then

\[
s_1 + (s_2 - s_1) + (s_3 - s_2) + \ldots + (s_k - s_{k-1}) + (n - s_k) = n
\]

gives a composition of \( n \) with \( k + 1 \) parts, namely \( x_1 = s_1 \), \( x_2 = s_2 - s_1 \), and so on until \( x_{k+1} = n - s_k \), and we observe that \( x_1 \geq 1, x_{k+1} \geq 0 \) whereas \( x_2, x_3, \ldots, x_k \geq 2 \). So we have a bijection \( f \) from sets \( \{s_1, s_2, \ldots, s_k\} \) to compositions \((x_1, x_2, \ldots, x_{k+1})\) of \( n \) with \( x_1 \geq 1, x_{k+1} \geq 0 \) and \( x_2, x_3, \ldots, x_{k-1} \geq 2 \). From our work on counting compositions, we know the answer is

\[
\binom{n - k + 1}{k}
\]

To confirm, if \( n = 5 \) and \( k = 2 \), we expect \( \binom{4}{2} = 6 \) subsets of \( \{1, 2, \ldots, 5\} \) of size \( k = 2 \): indeed, they are \( \{1, 3\}, \{2, 4\}, \{3, 5\}, \{1, 4\}, \{2, 5\}, \{1, 5\} \).
7 Exercises

In all the exercises below, \( n, k, x_1, x_2, \ldots, x_k \) are positive integers. The starred questions are substantially harder than the other questions.

- Four dice are thrown, where the first die is six sided, the second is eight sided, the third is twelve sided, and the fourth is twenty sided, and the four numbers which turn up are recorded in sequence. How many such sequences are there?
- How many sequences \((x_1, x_2, \ldots, x_k)\) are there if \(1 \leq x_i \leq i\) for \(i \in [k]\)?
- How many sets \(\{x_1, x_2, \ldots, x_k\}\) are there if \(x_i \in \{2i, 2i + 1\}\) for \(i \in [k]\)?
- How many sequences \((x_1, x_2, \ldots, x_k)\) have \(x_1 = 0\) and \(|x_{i+1} - x_i| \leq 1\) for \(i \in [k-1]\)?
- Arrange \(n\) points with equal spacing around a circle. In how many ways can we select \(k\) of the points so that no two consecutive points are not selected?
- Check that
  \[
  \binom{n}{n_1 n_2 \ldots n_k} = \binom{n}{n_1} \binom{n-n_1}{n_2} \binom{n-n_1-n_2}{n_3} \cdots \binom{n-n_1-n_2-\ldots n_{k-1}}{n_k}.
  \]
- How many ordered partitions of the set \([kn]\) into \(n\) parts of size \(k\) are there?
- How many unordered partitions of the set \([kn]\) into \(n\) parts of size \(k\) are there?
- Show that the number of ordered partitions of \([n]\) with 2 parts is \(2^n - 2\).
  - Show that the number of ordered partitions of \([n]\) with 3 parts is \(3^n - 3 \cdot 2^n + 3,\ n > 1\).
- How many compositions \((x_1, x_2, \ldots, x_k)\) of \(n\) are there where \(x_i \geq 2\)?
- How many compositions \((x_1, x_2, \ldots, x_k)\) of \(n\) are there where \(x_i \geq i^2\)?
- Which is bigger: the number of compositions of 73 into 13 parts or the number of ordered partitions of 73 into 13 parts?
- How many compositions of \(n\) with \(k\) parts are there if all the parts are positive even integers?
- How many sequences of positive integers add up to \(n\)?
- Find a bijection \(f\) from the set of subsets of \([n]\) of even size to the set of subsets of \([n]\) of odd size. Consider the case that \(n\) is even separately from the case that \(n\) is odd.