

Cycle lengths in sparse graphs

Benny Sudakov ^{*} Jacques Verstraëte [†]

Abstract

Let $C(G)$ denote the set of lengths of cycles in a graph G . In the first part of this paper, we study the minimum possible value of $|C(G)|$ over all graphs G of average degree d and girth g . Erdős [7] conjectured that $|C(G)| = \Omega(d^{\lfloor (g-1)/2 \rfloor})$ for all such graphs, and we prove this conjecture. We also show that this is a lower bound for the number of odd cycle lengths in a graph of chromatic number d and girth g . Further results are obtained for the number of cycle lengths in H -free graphs of average degree d .

In the second part of the paper, motivated by the conjecture of Erdős and Gyárfás [8] (see also Erdős [9]) that every graph of minimum degree at least three contains a cycle of length a power of two, we prove a general theorem which gives an upper bound on the average degree of an n -vertex graph with no cycle of even length in a prescribed infinite sequence of integers. For many sequences, including the powers of two, our theorem gives the upper bound $e^{O(\log^* n)}$ on the average degree of graph of order n with no cycle of length in the sequence, where $\log^* n$ is the number of times the binary logarithm must be applied to n to get a number which is at most one.

1 Introduction

For a graph G , let $C(G)$ denote the set of integers whose elements are lengths of cycles in G . The study of cycles in graphs has long been fundamental, and many questions about properties of graphs that guarantee some particular range of cycle length have been considered. The central goal in this paper is to obtain lower bound on $|C(G)|$ when G is a graph of average degree d and girth g , or G is an H -free graph, and to determine which integers are guaranteed to appear in $C(G)$ under density conditions on G .

In the case of dense graphs, there are many results which determine when $C(G)$ is an interval, or almost an interval. One of the first results in this direction was obtained by Bondy [5], who proved that for every n -vertex graph G of minimum degree larger than $\frac{n}{2}$, $C(G) = \{3, 4, \dots, n\}$. Once the minimum degree of a graph is allowed to pass below $\frac{n}{2}$, we can no longer guarantee any odd cycles, as the graph may be bipartite. Also, one cannot guarantee a hamiltonian cycle, (equivalently, $n \in C(G)$). In this situation, the natural question is to ask when $C(G)$ contains all even integers up to 2ℓ , where 2ℓ is the length of a longest even cycle of G . Bollobás and

^{*}Department of Mathematics, Princeton University, Princeton, NJ 08544, and Institute for Advanced Study, Princeton. E-mail: bsudakov@math.princeton.edu. Research supported in part by NSF CAREER award DMS-0546523, NSF grant DMS-0355497, USA-Israeli BSF grant, Alfred P. Sloan fellowship, and the State of New Jersey.

[†]Department of Combinatorics and Optimization, University of Waterloo, Waterloo, ON N2L 3G1, Canada. E-mail: jverstra@math.uwaterloo.ca

Thomason [4] showed that this is the case if G has order n and size at least $\lfloor \frac{n^2}{4} \rfloor - n + 59$. The best result on this problem is by Gould, Haxell, and Scott [12], who proved that if an n -vertex graph G has minimum degree at least cn , where $c > 0$ is a constant, then G contains all even integers up to $2\ell - K$ for some constant K depending only on c . It is conjectured that for some constant $c > 0$, every hamiltonian n by n bipartite graph of minimum degree at least $c\sqrt{n}$ contains cycles of all even lengths in $\{4, 6, 8, \dots, 2n\}$. All of the above mentioned results are for dense graphs – graphs whose average degree is linear in the order of the graph. In this paper, we are interested in studying $C(G)$ for sparse graphs.

Since n -vertex graphs of average degree d may have girth at least $\log_{d-1} n$, it is clear that for sparse graphs one cannot hope to state that $C(G)$ contains any integer from a finite set. Erdős and Hajnal [8] conjectured

$$\sum_{\ell \in C(G)} \frac{1}{\ell} = \Omega(\log d)$$

whenever G has average degree d . This conjecture was proved by Gyárfás, Komlós and Szemerédi [14]. Their result shows that if a graph does not have too many short cycles, then it must have many long cycles. However, it appears to be very difficult to pass from such statement to statements about the size or arithmetic structure of $C(G)$. For example, when does $C(G)$ contain a power of two, and how large is $|C(G)|$ when G has girth g ?

1.1 Cycles in graphs of large girth

The first problem we study is to determine the size of $C(G)$ for graphs of given average degree and girth (the *girth* of G is the length of the shortest cycle in G). The length of a longest cycle in a graph of girth g was first studied by Ore [20]. For graphs of girth at most four and average degree d , it is straightforward to prove that the longest cycle has length at least $d + 1$ if the girth is three, and at least $2d$ if the girth is four, and the proofs of these facts show $|C(G)| \geq d - 1$, with equality for K_{d+1} and $K_{d,d}$. Erdős [7] conjectured that $|C(G)| = \Omega(d^{\lfloor (g-1)/2 \rfloor})$ whenever G has average degree d and girth g . This was proved for $g = 5$ by Erdős, Faudree, Rousseau and Schelp [10]. They also show that $|C(G)| = \Omega(d^{5/2})$ for $g = 7$, $|C(G)| \geq \Omega(d^3)$ for $g = 9$ and $|C(G)| = \Omega(d^{g/8})$ in general. For comparison, Erdős' conjecture is $|C(G)| = \Omega(d^3)$ for $g = 7$ and $|C(G)| = \Omega(d^4)$ for $g = 9$. In Section 2, we will give a short proof of Erdős' conjecture. In fact, we obtain the following stronger theorem.

Theorem 1.1 *Let G be a graph of average degree d and girth g . Then $C(G)$ contains $\Omega(d^{\lfloor (g-1)/2 \rfloor})$ consecutive even integers.*

To see why Theorem 1.1 cannot be improved, recall that the *Moore Bound* for a graph G of minimum degree d and girth g states

$$|V(G)| \geq \begin{cases} 1 + d + d(d-1) + \dots + d(d-1)^{\lfloor \frac{g-1}{2} \rfloor - 1} & \text{if } g \text{ is odd} \\ 2\left(1 + (d-1) + (d-1)^2 + \dots + (d-1)^{\lfloor \frac{g-1}{2} \rfloor}\right) & \text{if } g \text{ is even} \end{cases}$$

Up to the constant factor, it is known that this bound is tight for infinitely many values of d whenever $g \leq 8$ or $g = 12$ and is also believed that for all other values of g there are graph with girth g and order $O(d^{\lfloor (g-1)/2 \rfloor})$. So it is evident that $|C(G)| = O(d^{\lfloor (g-1)/2 \rfloor})$ for such graphs. A related problem is to determine the number of odd integers in $C(G)$ when G has large chromatic number and girth. For example, Gyárfás [13] proved that a graph of chromatic number at least $2d+1$ contains cycles of d distinct odd length, and equality holds only for trees of complete graphs. Using similar techniques as in proof of Theorem 1.1 we can generalize the result of Gyárfás as follows: if G is a graph of chromatic number d and girth g , then $C(G)$ contains $\Omega(d^{\lfloor (g-1)/2 \rfloor})$ consecutive integers.

1.2 Cycles in H -free graphs

A graph is H -free if it contains no subgraph isomorphic to H . We consider the following generalization of Theorem 1.2 in Section 3. Given a bipartite graph H , determine a lower bound for $|C(G)|$ when G is an H -free graph of average degree d . Specifically, we consider r -half-bounded bipartite graphs. A bipartite graph is r -half-bounded if the degrees of all the vertices in one color class are at most r . An example of such graph is a complete bipartite graph $K_{r,s}$ with parts of size $r \leq s$. Using recent estimates on Turán numbers for r -half-bounded graphs due to Alon, Krivelevich and Sudakov [1], we prove the following result.

Theorem 1.2 *Let H be a fixed bipartite graph containing a cycle and let G be an H -free graph of average degree d . Then there exists a constant $t > 1$ depending on H such that $C(G)$ contains $\Omega(d^{t/(t-1)})$ consecutive even integers. Furthermore, we can take $t = r$ if H is r -half-bounded, and $t = 1 + \frac{1}{k-1}$ if H is a $2k$ -cycle.*

Notice that when H is a $2k$ -cycle, this result generalizes our Theorem 1.1 from graphs of girth $2k+1$ or $2k+2$ to graphs with no $2k$ -cycle. The estimate for r -half-bounded graphs in Theorem 1.2 is tight for every value of $r \geq 2$. Indeed, by the construction of projective norm graphs in [2] (modifying that in [15]) for every fixed $s \geq (r-1)! + 1$ there are graphs of order $O(d^{r/(r-1)})$ and average degree d which do not contain copy of $K_{r,s}$.

1.3 Arithmetic structure of $C(G)$

In the second part of the paper, we discuss the arithmetic structure of $C(G)$ for sparse graphs. The type of question we would like to answer is: what is the smallest d such that every graph of average degree at least d has a cycle of length equal to a square, or a power of two, or twice a prime? Our main theorem is motivated by the conjecture of Erdős and Gyárfás [8], stating that every graph of minimum degree at least three contains a cycle of length a power of two, and by the questions posed by Erdős (see page 228 of [9]). Throughout this section, for a sequence $(\sigma(i))_{i \geq 1}$ of integers, a σ -cycle is a cycle of length $\sigma(i)$ for some $i \geq 1$. We write $\pi < \sigma$ to denote that π is a subsequence of σ .

Perhaps the most natural starting point is to determine when $C(G)$ contains an integer congruent to zero modulo a given integer k . The first result in this direction was proved by Bollobás [3], who showed that if G has average degree at least $\frac{2}{k}(k+1)^k$, then G contains a cycle of length zero modulo k . The main result in [22] (see also Fan [11]) shows that if σ is any infinite increasing sequence of even integers such that $|\sigma(j) - \sigma(j-1)| \leq k$ for all $j \geq 1$, then every graph of average degree at least $4k$ contains a σ -cycle. In this section, we are interested in extending this result to the case that $|\sigma(j) - \sigma(j-1)|$ is not bounded. The theorem below gives an upper bound on the average degree of a graph containing no σ -cycles.

Theorem 1.3 *For any infinite increasing sequence σ of positive even integers and for any n -vertex graph G , if G contains no σ -cycle, then G has average degree at most*

$$\inf_{\substack{\pi \leq \sigma \\ r \geq 1}} \exp \left(6r + \sum_{i=1}^r \frac{2 \log \Delta(i)}{\pi(i-1)} + \frac{2 \log n}{\pi(r)} \right),$$

where $\pi(0) := 1$, $\Delta(1) := \pi(1)$, and $\Delta(i) = \max\{\sigma(j) - \sigma(j-1) : \sigma(j) \leq \pi(i)\}$ for $i \geq 2$.

To illustrate this statement consider the case when $\sigma(i) = 2^i$ for $i \geq 1$. Then we can take π to be the sequence of towers of twos, namely

$$\pi(1) = 2 \quad \pi(2) = 2^2 \quad \pi(3) = 2^{2^2} \quad \dots \quad \pi(i) = 2^{2^{\dots^2}}$$

so that $\pi(i) = 2^{\pi(i-1)}$, and take $r = \log^* n$, where $\log^* n = i$ whenever $\pi(i-1) < n \leq \pi(i)$. Then Theorem 1.3 implies that every graph of order n with no cycle of length a power of two has average degree $e^{O(\log^* n)}$. In fact, the same bound holds for many sequences, such as twice primes, squares, and the tower sequence π defined above. We say that a sequence σ is *exponentially bounded* if there is an absolute constant $C > 1$ such that $\sigma(i) \leq C\sigma(i-1)$ for all $i \geq 2$.

Corollary 1.4 *Let σ denote an infinite increasing exponentially bounded sequence of positive even integers. Then any n -vertex graph with no σ -cycles has average degree $e^{O(\log^* n)}$.*

In Section 4, we will construct sequences σ and n -vertex graphs with no σ -cycles whose average degrees have the same order of magnitude as the upper bound in Theorem 1.3, up to absolute constant factors in the exponent. However, these sequences are not exponentially bounded. It would be interesting to see if there are graphs with arbitrarily large average degree containing no σ -cycles, for some exponentially bounded sequence σ .

2 Cycles in graphs of large girth

The (open) neighborhood of $X \subset V(G)$ in a graph G is defined by

$$\partial X = \{y \in V(G) \setminus X \mid \exists x \in X : \{x, y\} \in E(G)\}.$$

In words, this is the set of vertices not in X and adjacent to at least one vertex of X . The d -core of a graph G , when it exists, is the subgraph obtained by repeatedly deleting vertices of degree

at most $d - 1$. It is a well-known fact that if a graph has integer average degree $2d$, then it has a d -core. It is convenient to assume throughout that d is an integer. Our first lemma states that graphs of large average degree and girth expand on small sets.

Lemma 2.1 *Let G be a graph of girth g and minimum degree at least $6(d + 1)$. Then, for every $X \subset V(G)$ of size at most $\frac{1}{3}d^{\lfloor (g-1)/2 \rfloor}$,*

$$|\partial X| > 2|X|.$$

Proof. Suppose $|\partial X| \leq 2|X|$ for some $X \subset V(G)$. Let H be the subgraph of G spanned by the set $Y = X \cup \partial X$. Then $|Y| \leq 3|X|$ and $e(H) \geq 6(d + 1)|X| \geq 2(d + 1)|Y|$. Thus H contains a subgraph Γ with minimum degree $d + 1$. Applying the Moore Bound to Γ , we obtain:

$$3|X| \geq |Y| \geq |V(\Gamma)| \geq 1 + (d + 1) \sum_{i < \lfloor (g-1)/2 \rfloor} d^i > d^{\lfloor (g-1)/2 \rfloor}$$

and therefore $|X| > \frac{1}{3}d^{\lfloor (g-1)/2 \rfloor}$, as required. ■

Using Lemma 2.1, we prove the conjecture of Erdős stating $|C(G)| = \Omega(d^{\lfloor (g-1)/2 \rfloor})$ when G has girth g and average degree d . A key ingredient of the proof is a lemma of Pósa [21] (see also [17], Exercise 10.20) which says that if G is a graph and $|\partial X| > 2|X|$ for every $X \subset V(G)$ of size at most m , then G contains a path of length $3m$.

Theorem 2.2 *For any graph G of girth g and average degree $48(d + 1)$, $|C(G)| \geq \frac{1}{8}d^{\lfloor (g-1)/2 \rfloor}$.*

Proof. Let H be a maximum bipartite subgraph of G , containing at least half of the edges of G . Then some connected component F of H has average degree at least $24(d + 1)$. Let T be a breadth first search tree in F , and let L_i denote the set of vertices of T at distance i from the root of T . Since F is bipartite, no edge of F joins two vertices of L_i . Denote by $e(L_i, L_{i+1})$ the number of edges of F with one endpoint in L_i and one endpoint in L_{i+1} . Then

$$\begin{aligned} \sum_i e(L_i, L_{i+1}) &= e(F) \geq 12(d + 1)|V(F)| = 12(d + 1) \sum_i |L_i| \\ &= 6(d + 1) \sum_i (|L_i| + |L_{i+1}|). \end{aligned}$$

Thus, there is an index i such that the subgraph $F_i \subset F$ induced by $L_i \cup L_{i+1}$ has average degree at least $12(d + 1)$. Then F_i contains a subgraph Γ with minimum degree $6(d + 1)$. By Lemma 2.1 we have that $|\partial X| > 2|X|$ for every $X \subset V(\Gamma)$ of size at most $\frac{1}{3}d^{\lfloor (g-1)/2 \rfloor}$. Hence Γ contains a path P of length $d^{\lfloor (g-1)/2 \rfloor}$ by Pósa's Lemma. Let T' be a minimal subtree of T whose set of end vertices is exactly $V(P) \cap L_i$. The minimality of T' ensures that it branches at the root. Let A be the set of vertices in $V(P) \cap L_i$ in one of these branches and let $B = (V(P) \cap L_i) \setminus A$. Then both A, B are nonempty and all paths from A to B through the root of T' have the same length, say $2h$. We may assume that $|B| \geq |A| \geq \frac{1}{4}|P|$ and let a be an arbitrary vertex in A . Then there are at least $\frac{1}{2}|B| \geq \frac{1}{8}|P|$ vertices of B on the same side of path from a . Hence there are subpaths of P from a to a vertex of B of at least $\frac{1}{8}|P|$ different lengths. For any such path Q , there is a unique subpath R of T' through the root joining the endpoints of Q , so that $Q \cup R$ is a cycle in G . Since all R have the same length $2h$, we obtain $\frac{1}{8}d^{\lfloor (g-1)/2 \rfloor}$ cycles of different lengths, and $|C(G)| \geq \frac{1}{8}d^{\lfloor (g-1)/2 \rfloor}$. ■

2.1 Proof of Theorem 1.1

To obtain Theorem 1.1, we will slightly modify the proof of Theorem 2.2. A θ -graph is a graph consisting of three internally disjoint paths between two vertices. We observe the following lemma as a corollary of the proof of Theorem 2.2:

Lemma 2.3 *Let G be a graph of average degree $48(d+1)$ and girth g . Then G contains a θ -graph containing a cycle of length at least $d^{\lfloor (g-1)/2 \rfloor} + 2$.*

Proof. Let the path P and the tree T' be defined as in the proof of the preceding theorem. If R is a path in T' joining the endpoints of P , then $P \cup R$ is a cycle of length at least $d^{\lfloor (g-1)/2 \rfloor} + 2$. Finally, for some path $Q \subset T'$ from the root of T' to an interior vertex of P , the subgraph $P \cup Q \cup R$ is the required θ -graph. ■

It is convenient to define an AB -path in a graph G to be a path with one endpoint in A and one endpoint in B , where $A, B \subset V(G)$. The following result of Bondy and Simonovits [6] (see also [22]) will be used to prove Theorem 1.1.

Lemma 2.4 *Let Γ be a θ -graph and let (A, B) be a nontrivial partition of $V(\Gamma)$. Then Γ contains AB -paths of all lengths less than $|V(\Gamma)|$ unless Γ is bipartite with bipartition (A, B) .*

Proof of Theorem 1.1. Let G be a graph of average degree $192(d+1)$ and girth g and let H be a maximum bipartite subgraph of G . Then some connected component F of H has average degree at least $96(d+1)$. Let T be a breadth-first search tree in F , and let L_i denote the set of vertices of T at distance i from the root. Then, for some i , the subgraph F_i of F induced by $L_i \cup L_{i+1}$ has average degree at least $48(d+1)$. By Lemma 2.3, F_i contains a θ -graph Γ containing a cycle of length at least $d^{\lfloor (g-1)/2 \rfloor} + 2$. Let T' be the minimal subtree of T whose set of end vertices is $V(\Gamma) \cap L_i$. Then there is a partition (A, B^*) of $V(\Gamma) \cap L_i$ such that all AB^* -paths in T' go through the root and have the same length, say $2h$. Let $B = V(\Gamma) \setminus A$. By Lemma 2.4, there exist AB -paths in Γ of all even lengths in $\{1, 2, \dots, d^{\lfloor (g-1)/2 \rfloor} + 2\}$. Since they have an even length, each such path is actually an AB^* -path, and the union of this path with the unique subpath of T' of length $2h$ joining its endpoints is a cycle. Therefore $C(G)$ contains $d^{\lfloor (g-1)/2 \rfloor}$ consecutive even integers, as required. ■

2.2 Chromatic number and cycle lengths

Using the above methods, we prove that in a graph G of large chromatic number and girth, $C(G)$ contains long interval of consecutive integers. We only sketch the details, as they resemble the proof of Theorem 1.1.

Theorem 2.5 *Let G be a graph of chromatic number d and girth g . Then $C(G)$ contains $\Omega(d^{\lfloor (g-1)/2 \rfloor})$ consecutive integers.*

Proof. Take a breadth first search tree T in a component of G of chromatic number d , and let L_i denote the set of vertices at distance i from the root of T . Then, for some i , the subgraph of G induced by L_i has chromatic number at least $\frac{1}{2}d$. This subgraph contains a graph F of minimum degree at least $\frac{1}{2}d - 1$ (a minimal $\frac{1}{2}d$ -chromatic subgraph has this property). Now as in the proof of Theorem 1.1, F contains a θ -graph Γ containing a cycle of length $\Omega(d^{\lfloor (g-1)/2 \rfloor})$. Moreover here we can make sure that this cycle has odd length. If T' is a minimal subtree of T whose set of end vertices is $V(\Gamma)$, then T' branches at its root, and this gives a partition (A, B) of $V(\Gamma)$. By Lemma 2.4, there are AB -paths of all lengths less than $|V(\Gamma)|$, and these paths together with subpaths of T' give the required cycle lengths. \blacksquare

3 Cycles in H -free graphs

To prove Theorems 1.2 and 1.3, we will use the following lemma, which summarizes the ideas of Section 2. Recall that a property of graphs is *monotone* if it is closed under taking subgraphs. The *radius* of graph G is the smallest integer r for which there is a vertex v in G such that the distance from any other vertex of G to v is at most r .

Lemma 3.1 *Let \mathcal{P} be a monotone property of graphs, and suppose that for every graph $G \in \mathcal{P}$ with minimum degree d , and every set $X \subseteq V(G)$ of size at most $f(d)$,*

$$|\partial X| > 2|X|.$$

Then every $G \in \mathcal{P}$ of average degree at least $16d$ contains cycles of $3f(d)$ consecutive even lengths, the shortest having length at most twice the radius of G .

Proof. Let G' be a maximum bipartite subgraph of G , and let T be a breadth-first search tree in a connected component F of G' of average degree at least $8d$. If L_i is the set of vertices at distance i from the root of T in F , then for some i , the subgraph F^* of F induced by $L_i \cup L_{i+1}$ has average degree at least $4d$. Let T^* be a breadth first search tree in a connected component of F^* of average degree at least $4d$. If L_j^* is the set of vertices at distance j from the root of T^* in F^* , then for some j , the subgraph F_j^* of F^* induced by $L_j^* \cup L_{j+1}^*$ has average degree at least $2d$. Now let Γ be a subgraph of F_j^* with minimum degree at least d . Since \mathcal{P} is monotone property and $G \in \mathcal{P}$ we have that also $\Gamma \in \mathcal{P}$. Therefore, $|\partial X| > 2|X|$ for every subset $X \subset V(\Gamma)$ of size at most $f(d)$. By Pósa's Lemma, there is a path $P \subset \Gamma$ of length $3f(d)$. If T' is a minimal subtree of T^* whose set of end vertices is $V(P) \cap L_j^*$, then as in Lemma 2.3, $P \cup T'$ contains a θ -graph, J , containing a cycle of length $3f(d) + 2$. Let T'' be the minimal subtree of T whose set of end vertices is $V(J)$. Applying Lemma 2.4 as in the proof of Theorem 1.1, we see that $J \cup T''$ contains cycles of $3f(d)$ consecutive even lengths in G . Since the shortest cycle has length at most $2i + 2$, the proof is complete. \blacksquare

In what follows, we denote by $\text{ex}(n, H)$ the *Turán number* of graph H , which is the maximum number of edges in an H -free graph on n vertices. To obtain expansion in H -free graphs, we show that it is enough to find upper bounds for $\text{ex}(n, H)$.

Lemma 3.2 *Let $a > 0$, $\frac{1}{2} < b < 1$ be real numbers such that for any positive integer n ,*

$$\text{ex}(n, H) \leq an^{2b}.$$

Then, for any H -free graph G of minimum degree at least $18ad$, and any subset X of vertices of G of size at most $d^{1/(2b-1)}$, $|\partial X| > 2|X|$.

Proof. Suppose that X is a subset of G of size m such that $|\partial X| \leq 2|X|$. Let G_Y be the subgraph of G induced by $Y = X \cup \partial X$. Then $|Y| \leq 3m$ and $e(G_Y) \geq 9ad|X| = 9adm$. On the other hand, since G_Y is H -free we have that

$$9adm \leq e(G_Y) \leq \text{ex}(|Y|, H) \leq a|Y|^{2b} \leq a(3m)^{2b} < 9am^{2b}.$$

Therefore $m^{2b-1} > d$, which proves the lemma. ■

Proof of Theorem 1.2. By the well known result of Kövari, Sós and Turán [16], for every bipartite graph H there are two constants $t > 1$ and c depending only H such that

$$\text{ex}(n, H) \leq cn^{2-1/t}.$$

By Lemma 3.2, with $a = c$ and $b = 1 - 1/2t$, every H -free graph F of minimum degree at least $18cd$ has the property that for every $X \subseteq V(F)$ of size at most $f(d) = d^{t/(t-1)}$, $|\partial X| > 2|X|$. By Lemma 3.1, with \mathcal{P} equal to the set of all H -free graphs, we deduce that every $G \in \mathcal{P}$ of average degree $288cd$ contains $3f(d)$ cycles of consecutive even lengths, proving the theorem. For the particular case when H is r -half-bounded, Alon, Krivelevich and Sudakov [1] showed that

$$\text{ex}(n, H) = O(n^{2-\frac{1}{r}}),$$

so the proof above applies with $b = 1 - 1/(2r)$ and gives $\Omega(d^{r/(r-1)})$ cycles of consecutive even lengths. Finally, if $H = C_{2k}$, then Corollary 9 in [22] shows

$$\text{ex}(n, H) = 8kn^{1+\frac{1}{k}},$$

so we can apply the above proof with $b = \frac{1}{2} + \frac{1}{2k}$ to conclude that for every C_{2k} -free graph G , $C(G)$ contains $\Omega(d^k)$ consecutive even integers. ■

4 Arithmetic structure of $C(G)$.

Fix $\pi < \sigma$ and $r \geq 1$, and let \mathcal{P}_i denote the monotone property of graphs containing no cycle of length $\sigma(j)$ for all $\sigma(j) \leq \pi(i)$, and recall $\Delta(i) = \max\{\sigma(j) - \sigma(j-1) : \sigma(j) \leq \pi(i)\}$. To prove Theorem 1.3, we first prove the following claim.

Claim 4.1 For every n -vertex graph $G \in \mathcal{P}_r$,

$$e(G) \leq a_r n^{1 + \frac{2}{\pi(r)}},$$

where $a_1 = 4\pi(1)$ and a_i for $i \geq 2$ is any positive real satisfying

$$\pi(i-1) \log \frac{a_i}{288a_{i-1}} \geq 2 \log \Delta(i).$$

Proof. We proceed by induction on i . For $i = 1$, Corollary 9 in [22] gives

$$e(G) \leq 4\pi(1)n^{1 + \frac{2}{\pi(1)}},$$

and this proves the claim for $i = 1$. Suppose we have proved the claim for $j < i$, and let $G \in \mathcal{P}_i$ be an n -vertex graph with $e(G) > a_i n^{1+2/\pi(i)}$, where a_i satisfies the bounds in the claim. By the induction hypothesis we have that every m -vertex graph in \mathcal{P}_{i-1} has at most $a_{i-1} m^{1+2/\pi(i-1)}$ edges. Therefore by Lemma 3.2, we have that for any graph $F \in \mathcal{P}_{i-1}$ with minimum degree d every subset $X \subset V(F)$ of size at most

$$f(d) = \left(\frac{d}{18a_{i-1}} \right)^{\frac{1}{2}\pi(i-1)}$$

has $|\partial X| > 2|X|$.

Since G has n vertices and $e(G) \geq a_i n^{1+2/\pi(i)}$, by Lemma 6 in [22], there is a subgraph Γ of G of average degree at least a_i and radius at most $\frac{1}{2}\pi(i)$. Note that Γ has property \mathcal{P}_i and thus has also property \mathcal{P}_{i-1} . By Lemma 3.1, Γ contains cycles of at least $3f(\frac{a_i}{16})$ consecutive even lengths, the shortest of which has length at most $\pi(i)$. Since $\Gamma \in \mathcal{P}_i$, there must be less than $\Delta(i)$ of these consecutive even lengths, otherwise Γ contains a cycle of length $\sigma(j)$ for some $j \leq \pi(i)$. Therefore

$$\left(\frac{a_i}{288a_{i-1}} \right)^{\frac{\pi(i-1)}{2}} < 3f\left(\frac{a_i}{16}\right) < \Delta(i)$$

which contradicts the bounds on a_i in the claim. ■

Proof of Theorem 1.3. Recall that $\pi(0) = 1$, $\Delta(1) = \pi(1)$ and let

$$a_r = 4\pi(1)(288)^{r-1} \prod_{i=2}^r \exp\left(\frac{2 \log \Delta(i)}{\pi(i-1)}\right) < \frac{1}{2} \exp\left(6r + \sum_{i=1}^r \frac{2 \log \Delta(i)}{\pi(i-1)}\right).$$

Since a_r satisfies the condition of the Claim 4.1, we have that estimate on the number of edges of G from this claim is valid for any r . Therefore the average degree of G is at most

$$\inf_{r \geq 1} 2a_r n^{2/\pi(r)} \leq \inf_{r \geq 1} \exp\left(6r + \sum_{i=1}^r \frac{2 \log \Delta(i)}{\pi(i-1)} + \frac{2 \log n}{\pi(r)}\right).$$

This bound is valid for any $\pi < \sigma$, so this completes the proof of Theorem 1.3. ■

Proof of Corollary 1.4. Since σ is exponentially bounded, $\sigma(i) \leq C\sigma(i-1)$ for all $i \geq 2$. Let $r = \log^* n$ and let $\pi < \sigma$ be chosen so that $2^{\pi(i-1)} \leq \pi(i) \leq (2C)^{\pi(i-1)}$ for all $i \geq 2$. Note that $\pi(r) \geq n$. Then, since $\Delta(i) \leq \pi(i)$, the upper bound in Theorem 1.3 is

$$\exp\left(6r + \sum_{i=1}^r \frac{2 \log \pi(i)}{\pi(i-1)} + \frac{2 \log n}{n}\right) \leq \exp(6r + 2r \log(2C) + 2) = e^{O(\log^* n)}.$$

This proves Corollary 1.4. ■

In conclusion, we show that the bound in Theorem 1.3 cannot be improved in general.

Construction. We construct a sequence of graphs G_1, G_2, G_3, \dots such that $|V(G_k)| = n_k - 2$, G_k is $(\alpha_k + 1)$ -regular, and G_k has girth larger than n_{k-1} , where n_k is even for all k and $n_0 := 2$. By known probabilistic and explicit constructions of small graphs of large girth (see, e.g., [18] and [19]), we may take $\log n_k = n_{k-1} \log \alpha_k$. Now let σ be defined by $\sigma(i) = n_i$. Since G_i has girth larger than n_{i-1} and G_i has order less than n_i , none of the graphs G_i have a σ -cycle. We choose α_i so that

$$\alpha_i \geq 2^{2^{2^i}}.$$

If we take $\pi = \sigma$ in Theorem 1.3, and $r = i$, then we have $\Delta(i) \sim n_i$ as $i \rightarrow \infty$, and also

$$\frac{\log \alpha_i}{\sum_{j < i} \log \alpha_j} \rightarrow \infty.$$

Also, as $j \rightarrow \infty$,

$$\frac{2 \log \Delta(j)}{\pi(j-1)} \sim \frac{2 \log n_j}{n_{j-1}} = 2 \log \alpha_j,$$

and the upper bound on the average degree of G_i from Theorem 1.3 is:

$$\exp\left(6i + \sum_{j=1}^i \frac{2 \log \Delta(j)}{\pi(j-1)} + \frac{2 \log n_i}{\pi(i)}\right) = \alpha_i^{2+o(1)}.$$

Since G_i has average degree $\alpha_i + 1$, the bound given by Theorem 1.3 is tight up to the constant factor in the exponent.

5 Concluding Remarks

- **Even cycles.** It would be interesting to determine if there is an infinite increasing exponentially bounded sequence σ for which there are graphs of arbitrarily large average degree containing no σ -cycles. Erdős [9] states that this is probably true when σ is the sequence of powers of two, although no example of a graph of minimum degree three with no cycle of length a power of two is known. The construction in Section 4 shows that if we take a sequence σ of positive integers defined by $\log \sigma(i) = 2^{2^i} \sigma(i-1)$, for $i \geq 1$, then there are graphs of arbitrarily large average degree with no σ -cycles.

- **Odd cycles.** Erdős [9] posed analogous questions for odd cycles in graphs of large chromatic number, for example, does every graph of infinite chromatic number contain a cycle of length equal to an odd integer square? By repeating a similar construction to that given in Section 4, it is possible to show that there are (very fast-growing) infinite increasing sequences of odd integers σ and graphs of infinite chromatic number containing no σ -cycle. On the other hand, we ask the following concrete question: does every graph of chromatic number at least four contain a cycle of length one more than a power of two? This seems to be a natural generalization of the Erdős-Gyárfás [8] conjecture.

- **Chromatic number and cycle lengths.** It seems that the result of Theorem 2.5 can be further improved. In particular in [7], Erdős asked whether for every $\epsilon > 0$ and sufficiently large d , every triangle free graph of chromatic number d contains at least $\Omega(d^{2-\epsilon})$ cycles of different length. More generally one can ask if every graph of girth at least $2t$ and chromatic number d contains at least $\Omega(d^{t-\epsilon})$ cycles of different length. We believe that the techniques which were developed in this paper may be useful to attack these problems.

References

- [1] N. Alon, M. Krivelevich and B. Sudakov, Turán numbers of bipartite graphs and related Ramsey-type questions, *Combinatorics Probability and Computing* 12 (2003), 477–494.
- [2] N. Alon, L. Rónyi and T. Szabó, Norm-graphs: variations and applications, *J. Combinatorial Theory Ser. B* 76 (1999), 280–290.
- [3] B. Bollobás, Cycles modulo k , *Bull. London Math. Soc.* 9 (1977), 97–98.
- [4] B. Bollobás and A. Thomason, Weakly pancyclic graphs, *J. Combinatorial Theory Ser. B* 77 (1999), 121–137.
- [5] A. Bondy, Pancyclic graphs I, *J. Combinatorial Theory Ser. B* 11 (1971) 80–84.
- [6] A. Bondy and M. Simonovits, Cycles of even length in graphs, *J. Combinatorial Theory Ser. B* 16 (1974), 97–105.
- [7] P. Erdős, Some of my favourite problems in various branches of combinatorics, *Matematiche (Catania)* 47 (1992), 231–240.
- [8] P. Erdős, Some of my favorite solved and unsolved problems in graph theory, *Quaestiones Math.* 16 (1993), 333–350.
- [9] P. Erdős, Some old and new problems in various branches of combinatorics. Graphs and combinatorics (Marseille, 1995). *Discrete Math.* 165/166 (1997), 227–231.
- [10] P. Erdős, R. Faudree, C. Rousseau and R. Schelp, The number of cycle lengths in graphs of given minimum degree and girth, *Discrete Mathematics* 200 (1999), 55–60.

- [11] G. Fan, Distribution of cycle lengths in graphs, *J. Combinatorial Theory Ser. B* 84 (2002), 187–202.
- [12] R. Gould, P. Haxell and A. Scott, A note on cycle lengths in graphs, *Graphs and Combinatorics* 18 (2002), 491–498.
- [13] A. Gyárfás, Graphs with k odd cycle lengths, *Discrete Mathematics* 103 (1992), 41–48.
- [14] A. Gyárfás, J. Komlós and E. Szemerédi, On the distribution of cycle lengths in graphs, *J. Graph Theory* 8 (1984), 441–462.
- [15] J. Kollár, L. Rónyai and T. Szabó, Norm-graphs and bipartite Turan numbers, *Combinatorica* 16 (1996), 399–406.
- [16] T. Kövari, V.T. Sós and P. Turán, On a problem of K. Zarankiewicz, *Colloquium Math.* 3 (1954), 50–57.
- [17] L. Lovász, **Combinatorial Problems and Exercises**, 2nd Ed., North-Holland, Amsterdam, 1993.
- [18] A. Lubotsky, M. Phillips and P. Sarnak, Ramanujan Graphs, *Combinatorica* 9 (1988), 261–277.
- [19] G. Margulis, Explicit group-theoretic constructions of combinatorial schemes and their applications in the construction of expanders and concentrators, *Problems Inform. Transmission* 24 (1988), 39–46.
- [20] O. Ore, On a graph theorem by Dirac. *J. Combinatorial Theory* 2 1967 383–392.
- [21] L. Pósa, Hamiltonian circuits in random graphs, *Discrete Mathematics* 14 (1976), 359–364.
- [22] J. Verstraëte, On arithmetic progressions of cycle lengths in graphs, *Combinatorics Probability and Computing* 9 (2000), 369–373.