Note on Vertex-Disjoint Cycles

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Abstract

Häggkvist and Scott asked whether one can find a quadratic function $q(k)$ such that if G is a graph of minimum degree at least $q(k)$, then G contains vertex-disjoint cycles of k consecutive even lengths. In this paper, it is shown that if G is a graph of average degree at least $k^2 + 19k + 12$ and sufficiently many vertices, then G contains vertex-disjoint cycles of k consecutive even lengths, answering the above question in the affirmative. The coefficient of k^2 cannot be decreased and, in this sense, this result is best possible.

All graphs considered in this paper are simple and finite. Throughout this paper, $K_{s,t}$ denotes the complete bipartite graph with s vertices in one colour class and t vertices in the other. Also, $|H|$ denotes the order of a graph H and $e(H)$ denotes the number of edges in H. The word *disjoint* will be taken to imply *vertex-disjoint*.

It is well-known that a graph of average degree at least two contains at least one cycle. From this it is easy to deduce that if k is a natural number, a graph of order n containing at least $n + k - 1$ edges contains k distinct cycles. Corradi and Hajnal [3] considered the requirement that the k cycles be disjoint. They proved that a graph of minimum degree at least $2k$ and order at least $3k$ contains k disjoint cycles. Under the same minimum degree condition, Egawa [4] showed that a graph of order at least $17k + o(k)$ contains k disjoint cycles of the same length.

In a recent paper by Bondy and Vince [2], it was shown that if G is a graph of order at least three containing at most two vertices of degree at most two, then G contains a pair of cycles of consecutive lengths or consecutive even lengths. The following general theorem was established by the author (see Theorem 1 in [12]):

Theorem 1 *Let k be a natural number and G a bipartite graph of average degree at least $4k$. Then there exist cycles of k consecutive even lengths in G. Moreover, the shortest of these cycles has length at most twice the radius of G.*

In this paper, we consider the associated problem of finding disjoint cycles of consecutive even lengths. Häggkvist and Scott [5] asked whether there exists a quadratic function $q(k)$ such that if G is a graph of minimum degree at least $q(k)$, then G contains disjoint cycles of k consecutive even lengths. We answer this question in the affirmative, by proving the following theorem:

Theorem 2 *Let k be a natural number and let G be a graph of order at least $n_k = (16k^2)!$ and average degree at least $k^2 + 19k + 12$. Then G contains disjoint cycles of k consecutive even lengths.*

Theorem 2 is best possible in the sense that for n large enough and $s = (k + 1)(k + 2)/2$, the complete bipartite graph $K_{s-1,n-(s-1)}$ does not contain k disjoint cycles of consecutive even lengths and has average degree at least $k^2 + 3k$. From the proof of Theorem 2, we may also deduce that a sufficiently large graph, of average degree at least $16k + 5$ and containing no cycle of length four, contains disjoint cycles of k consecutive even lengths. This type of difference between general graphs and graphs containing no cycle of length four was first observed by Häggkvist (see Thomassen [11], page 138). A similar phenomenon occurs for graphs of bounded maximum degree.

In order to prove Theorem 2, we first need two lemmas. The first lemma is a consequence of a lemma due to Kostochka and Pyber (see Lemma 1.1 page 84 in [7]):

Lemma 3 *Let r be a natural number, c a positive real number, and let G be a graph of order n and size at least $cn^{1+\frac{1}{r}}$. Then G contains a subgraph of average degree at least c and radius at most r.*

The second lemma is proved using a standard counting argument:

Lemma 4 *Let s and t be natural numbers and let G be a bipartite graph with colour classes A and B. If $|A| > t|B|^{s+1}$ and $e(G) \geq s|A|$, then G contains $K_{s,t}$.*

Proof. If $v \in A$ and the degree of v is $d(v)$, then there exist $\binom{d(v)}{s}$ subsets of B of size s in the neighbourhood of v. Since $|A| > t|B|^{s+1}$,

$$\begin{aligned}
\sum_{v\in A}\binom{d(v)}{s} &\geq \frac{1}{s^{s+1}}\sum_{v\in A} d(v)^s \\
&\geq \frac{1}{s^{s+1}}\cdot\frac{1}{|A|^{s-1}}\cdot\left(\sum_{v\in A} d(v)\right)^s \\
&\geq \frac{1}{s^{s+1}}\cdot\frac{1}{|A|^{s-1}}\cdot e(G)^s \\
&\geq \frac{1}{s^{s+1}}\cdot\frac{1}{|A|^{s-1}} s^s|A|^s \\
&\geq |A|/s > t|B|^s \geq t\binom{|B|}{s}.
\end{aligned}$$

So there exists a subset of B of size s contained in the intersection of the neighbourhoods of at least t vertices of A. This implies that $K_{s,t} \subset G$. □

Proof of Theorem 2. Let G be a graph of average degree at least $k^2 + 19k+12$, where $k \geq 2$. Suppose $G = G_0$ contains cycles of k consecutive even lengths. Define $r_0 = \min\{r : G$ contains cycles of lengths $2r, 2r+2, \ldots, 2r+2k-2\}$. Let X_0 be a subgraph of G_0 comprising the union of cycles of lengths $2r_0, 2r_0+2, \ldots, 2r_0+2k-2$ in G_0. Let $G_1 = G_0 - V(X_0)$. If $G_0, G_1, \ldots, G_i$ are defined, and G_i contains k cycles of consecutive even lengths, define $r_i = \min\{r : G_i$ contains cycles of lengths $2r, 2r+2, \ldots, 2r+2k-2\}$. Let

X_i be a subgraph of G_i comprising the union of cycles of lengths $2r_i, 2r_i + 2, \ldots, 2r_i + 2k - 2$ and let $G_{i+1} = G_i - V(X_0)$.

Note that the integers $r_0, r_1, r_2, \ldots$ form a non-decreasing sequence. If k of the integers r_i are identical, say $r_i = r_{i+1} = \ldots = r_{i+k-1} = m$, then $G = G_0$ contains k disjoint cycles of lengths $2m, 2m+2, \ldots, 2m+2k-2$, contained in $X_i, X_{i+1}, \ldots, X_{i+k-1}$ respectively. Therefore suppose no k of the integers r_i are identical.

We claim that $G = G_0$ contains cycles of k consecutive even lengths, the shortest of which has length at most $2(\log_2 n)^2$. Let B be a spanning bipartite subgraph of G_0 containing at least half of the edges of G_0. Set $r = \lfloor (\log_2 n)^2 \rfloor$. Since $n \geq n_k \geq 2^{k^2+1}$, $n^{1/r} \leq 1 + 1/2k$ and therefore

$$\begin{aligned} e(B) &\geq \frac{1}{4}(k^2 + 19k + 12) \\ &\geq \frac{1}{4}(16k + 8)n \\ &\geq 4kn^{1+\frac{1}{r}}. \end{aligned}$$

By Lemma 3, applied to B, there exists a subgraph H of B of average degree at least $4k$ and radius at most $r \leq (\log_2 n)^2$. By Theorem 1, H (and therefore G_0) contains cycles of k consecutive even lengths, the shortest of which has length at most $2(\log_2 n)^2$. This shows that $r_0 \leq (\log_2 n)^2$. Conversely, whenever G_i does not contain cycles of k consecutive even lengths, the shortest having length at most $2(\log_2 n)^2$, G_i has average degree at most $16k + 8$.

As G is finite, there exists a non-negative integer j such that $r_j \leq (\log_2 n)^2$ and G_{j+1} does not contain cycles of k consecutive even lengths, the shortest having length at most $2(\log_2 n)^2$. Let $G' = G_{j+1}$ and $G'' = G - V(G')$. Noting that $|X_i| \leq 2k(r_i + k - 2)$ and that no k of the integers r_i are identical,

$$|G''| \leq \sum_{i=0}^{j} |X_i|$$

$$\begin{aligned}
&\leq (k-1)\sum_{i=0}^{r} 2k(i+k-2) \\
&\leq (k-1)kr(r+1) + 2k(k-1)(k-2)(r+1) \\
&\leq k^3 r^2 \\
&\leq k^3(\log_2 n)^4.
\end{aligned}$$

As G' contains no cycles of k consecutive even lengths with shortest cycle of length at most $2(\log_2 n)^2$, G' has average degree at most $16k+8$. Let B' be the bipartite graph spanned by the edges of G with one end in G' and the other in G''. Then $e(B') = e(G) - e(G') - e(G'')$ and as $|G''|^2 \leq k^6(\log_2 n)^8 < n$,

$$\begin{aligned}
e(B') &\geq \frac{1}{2}(k^2+19k+12)n - \frac{1}{2}(16k+8)|G'| - e(G'') \\
&\geq \frac{1}{2}(k^2+19k+12)n - \frac{1}{2}(16k+8)n - |G''|^2 \\
&\geq \frac{1}{2}(k^2+3k+4)n - n \\
&\geq \frac{1}{2}(k^2+3k+2)n.
\end{aligned}$$

With $s = \frac{1}{2}(k^2+3k+2)$ and recalling that $n \geq n_k$,

$$\begin{aligned}
s|G''|^{s+1} &\leq sk^{3s+3}(\log_2 n)^{4s+4} \\
&< n - k^3(\log_2 n)^4 \\
&\leq n - |G''| = |G'|.
\end{aligned}$$

Applying Lemma 4 to B', we find a complete bipartite subgraph $K_{s,s}$ in B'. This is easily seen to contain k disjoint cycles of lengths $4, 6, \ldots, 2k+2$. $\square$

The linear term $19k+12$ in the statement of Theorem 2 can probably be improved. This theorem is best possible as seen by the complete bipartite graph in the example following the statement of Theorem 2. In light of this example, we make the following conjecture:

Conjecture 5 *Any graph of average degree at least $(k+1)(k+2)$ contains k disjoint cycles of consecutive even lengths.*

Given a graph H, a *topological* H is a graph obtained by arbitrarily subdividing the edges of H. We now turn to the problem of finding disjoint isomorphic topological H in a graph G.

This is a natural generalization of the problem of finding disjoint cycles of the same length in a graph G. In this direction, Egawa [4] showed that a sufficiently large graph of minimum degree at least $2k$ contains k disjoint cycles of the same length.

Mader [10] was the first to establish the existence of a constant c_t such that every graph of average degree at least c_t contains a topological complete graph of order t. Pyber and Kostochka [7] used Mader's result to show that dense graphs contain small topological complete graphs:

Theorem 6 *Let G be a graph of order n and size at least $4^{t^2} n^{1+\varepsilon}$, $\varepsilon > 0$. Then G contains a topological complete graph of order at most $7t^2(\log_2 t)/\varepsilon$.*

Given a graph H, we now apply the method of the proof of Theorem 2 to find many disjoint isomorphic topological H subgraphs in a graph of high enough average degree:

Theorem 7 *Let H be a graph of order h and let n, k be positive integers satisfying $\max\{kh, 2e(H)k^{kh+2}(8h^3 \log_2 n)^{(kh+1)e(H)}\} \leq n$. If G is a graph of order n and average degree at least $4^{h^2+1} + 2kh + 1$, then G contains k disjoint isomorphic topological H.*

Proof. If H is empty, then as $n \geq kh$ we easily find k disjoint isomorphic H in G. Therefore, suppose H is non-empty. The number of isomorphic topological H of order m is then at most the number of representations of $m - h$ as an ordered sum of $e(H)$ non-negative integers. This number is precisely

$$\binom{m - h + e(H) - 1}{e(H) - 1}.$$

Let $G = G_0$ and define $G_1 = G_0 - V(H_0)$ where H_0 is a smallest topological H appearing in G. By Theorem 5, with $\varepsilon = (\log_2 n)^{-1}$, we can guarantee that $|H_0| \leq 7h^2 \log_2 h \cdot \log_2 n = m$. In general, if G_{i-1} is defined, let $G_i = G_{i-1} - V(H_{i-1})$ where H_{i-1} is a smallest topological H appearing in G_{i-1}. We continue this procedure until we reach a stage j where G_j does not contain a topological H of order at most m. Let $G' = G_j$ and $G'' = G - V(G_j)$. If k of the H_i are isomorphic in G'' then the requirements of the theorem are met, so we suppose this is not the case. Then

$$\begin{aligned} |G''| &\leq \binom{m-h+e(H)-1}{e(H)-1}(k-1)m \\ &\leq km(m-h+e(H)-1)^{e(H)-1} \\ &\leq k(8h^2 \log_2 h \cdot \log_2 n)^{e(H)} \\ &\leq k(8h^3 \log_2 n)^{e(H)}. \end{aligned}$$

Under the assumption that $2e(H)k^{kh+2}(8h^3 \log_2 n)^{(kh+1)e(H)} \leq n$, the above inequality implies that $|G'| = n - |G''| > \frac{n}{2} \geq ke(H)|G''|^{kh+1}$.

By the definition of G'' and Theorem 5, $e(G') < 2 \cdot 4^{h^2}|G'|$. The number of edges in the bipartite graph B, spanned by the edges with one end in G' and the other end in G'', is therefore at least $\frac{1}{2}(4^{h^2+1} + 2kh - 4 \cdot 4^{h^2})|G'| \geq kh|G'|$. Applying Lemma 4 to B with $s = kh$ and $t = ke(H)$, we deduce that B contains $K_{kh,ke(H)}$. This complete bipartite graph contains k disjoint isomorphic topological H, in which every edge of H is subdivided precisely once. □

The expression $4^{h^2+1} + 2kh + 1$ in the statement of the above theorem is probably not best possible. Deep theorems of Bollobás and Thomason [1] and Komlós and Szemerédi [8] show that a graph of average degree ch^2, for some absolute contant $c > 0$, contains a topological complete graph of order h. It may be true that for a different value of c, we even get a pair of disjoint isomorphic topological complete graphs of order h.

It is not hard to see that the lower bound on n in the theorem can be improved. To obtain k vertex disjoint isomorphic topological H in a bipartite graph, we need only that it contain a large number of vertex disjoint complete bipartite graph rather than one large complete bipartite graph. The following Lemma makes this more precise:

Lemma 8 *Let H be a graph of independence number α and order h. Let m be the minimum number of vertices in the smaller colour class in a bipartite topological H, taken over all bipartite topological H. Then $m = h - \alpha$.*

Proof. By subdividing all the edges of H not incident with a maximum independent set, we see that $m \leq h - \alpha$. For each set of $\alpha + r$ vertices of H, the subgraph induced by these vertices has at least r edges, by definition of the independence number. So, if each of these edges has to be subdivided we must have at least r new vertices on these edges. This shows that $m \geq h - \alpha$. Hence $m = h - \alpha$. □

The remarks before the lemma, together with the same ideas as in the preceding theorem give the following result:

Theorem 9 *Let H be a graph of order h and independence number α. Let G be a graph of minimum degree at least $k(h - \alpha) + o(k)$ and order at least $kh^{4e(H)+1}$. Then G contains k disjoint isomorphic topological H.*

We notice that when $H = K_3$, we are precisely considering vertex-disjoint cycles of the same length, which have been investigated by a number of authors. This result is best possible up to the term $o(k)$, since by the preceding lemma, $K_{k(h-\alpha)-1,n}$ contains no k disjoint topological H. This is to be compared with the results arising from the Alon-Yuster conjecture: a number of authors have shown that we can cover most of the vertices of a graph G of order n and minimum degree at least $n/2$ with disjoint copies of a prescribed bipartite graph. Komlós (2000) has suggested that $n/2$ can be replaced by $(1 - \alpha/h)n$ when H has order h and independence number α. The question here is what happens for $kh \leq n \leq kh^{4e(H)+1}$. It is very likely that replacing the $o(k)$ term with a constant depending only on H is very difficult, as seen in the case of the Erdős-Faudree conjecture (tiling a graph of order n and minimum degree at least $n/2$ with 4-cycles).

Jørgensen and Pyber [6] have discussed problems involving covering the edges of a graph with topological subgraphs. We consider an analogue for covering vertices. Let H be a fixed graph. We let $f(r, H)$ denote the maximum proportion of vertices that can be covered with disjoint topological H in any r-regular graph. We define

$$\begin{aligned} f_{-}(H) &= \liminf_{r\to\infty} f(r, H) \\ f_{+}(H) &= \limsup_{r\to\infty} f(r, H). \end{aligned}$$

Observe that $1/2 \leq f_{-}(H) \leq f_{+}(H) \leq 1$ for every H. The lower bound follows by removing topological H from an r-regular graph G until we obtain a graph F containing no topological H. Then the average degree of F is bounded above by a constant $c > 0$ depending only on H, by the results of Mader [10]. As G is r-regular, the number of edges between F and the remainder of G is at most $r(|G| - |F|)$, but at least $(r - c)|F|$. Therefore $|F| \leq \frac{1}{2}(1 - \frac{c}{r})^{-1}|G|$ and hence $f_{-}(H) \geq \frac{1}{2}$.

When H is a complete graph of order two or three, the problem above becomes the problem of covering with paths or cycles respectively. Petersen's 2-factor theorem (see [9] page 54) states that for $k \geq 1$, every $2k$-regular graph contains a 2-factor. This shows that if H has order at most three, then $f_{+}(H) = 1$. The author does not know of any graph H for which $f_{-}(H) < 1$ or $f_{-}(H) < f_{+}(H)$; this problem may prove to be very interesting.

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