Note – Eulerian tours

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An eulerian tour in a graph is a closed walk which passes exactly once through every edge of the graph. A graph is eulerian if it has an eulerian tour. A fundamental question is, given a graph G, how can we tell if it has an eulerian tour or hamiltonian cycle? We will answer the question for a more general class of objects called pseudographs. A pseudograph is a pair (V, E) where V is a set of vertices and E is a set of unordered lists of two elements of V. A pseudograph can have many edges between two vertices as well as many edges on a single vertex – we refer to these as loops. The degree of a vertex in a pseudograph is $2\ell + m$ where ℓ is the number of loops on the vertex and m is the number of edges on the vertex which are not loops. In other words, loops contribute two to the degree of a vertex. As an example, the pseudograph below has the given degrees:

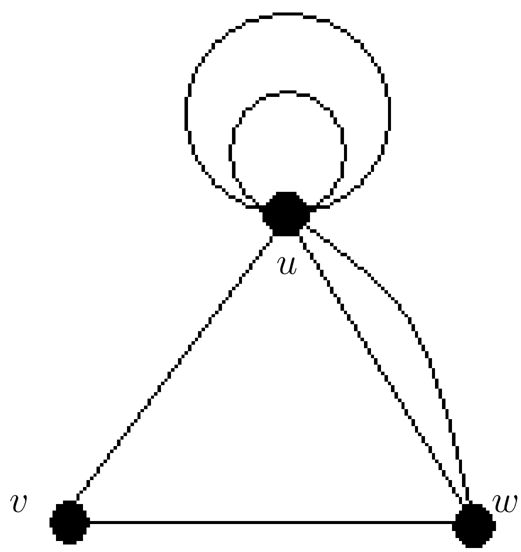

Figure : $d(u) = 7$, $d(v) = 2$, $d(w) = 3$.

We will prove that every connected pseudograph where all degrees are even has an eulerian tour. The use of pseudographs actually makes the proof easier.

Theorem 1 *A pseudograph G is eulerian if and only if it is connected and all its vertices have positive even degree.*

Proof ▷ If G is eulerian, then clearly G is connected since there is a walk between any two vertices of G. Furthermore, every vertex of G has even degree since an eulerian tour contributes two edges at a time at each vertex: every time the tour visits a vertex, there is an incoming edge and an outgoing edge from that vertex. Now suppose G is a connected pseudograph and all vertices of G have even degree. We want to show that G has an eulerian tour, and we will prove it by induction on the number n of edges of G. The base case is $n = 1$: then we have a vertex with a loop on it, which clearly is eulerian. In fact we can assume G has at least two vertices otherwise G consists of one vertex with loops on it which is eulerian.

Now suppose $n > 1$ and that every graph with $n - 1$ edges and at least two vertices is eulerian. Let G be a graph with n edges. Pick a vertex $v \in V(G)$ and two edges $e = \{v, u\}$ and $f = \{v, w\}$. We can make sure that those edges are not loops, since G is connected with $n > 1$ edges and the degree of v is even. Remove e and f from G and add $\{u, w\}$. This operation is shown below:

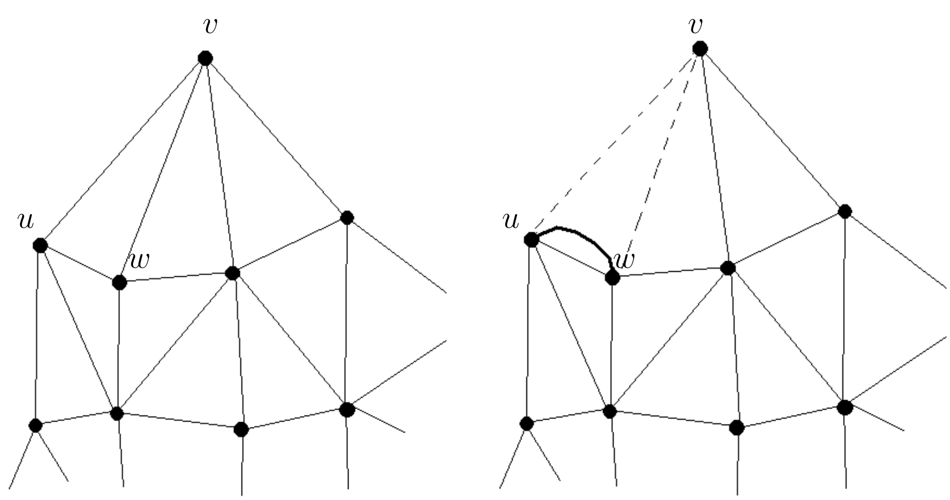

Figure : operation for eulerian tours.

Then all vertices of this graph H have even degree and $|E(H)| = n - 1$. If H is connected, let $F = H$. If H is not connected, then v is an isolated vertex[1] in H and let $F = H - v$. Since F is connected, F has an eulerian tour by induction which passes through the edge $\{u, w\}$ – the bold edge in the figure. Remove $\{u, w\}$ from the tour and add $\{u, v\}$ and $\{v, w\}$ to get an eulerian tour in G. ∎

[1]Try to prove that this is the only way H could be disconnected.