Even Cycles in Hypergraphs.

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Abstract

A cycle in a hypergraph $\mathcal{A}$ is an alternating cyclic sequence $A_0, v_0, A_1, v_1, \ldots, A_{k-1}, v_{k-1}, A_0$ of distinct edges A_i and distinct vertices v_i of $\mathcal{A}$ such that $v_i \in A_i \cap A_{i+1}$ for all i modulo k. In this paper, we determine the maximum number of edges in hypergraphs on n vertices containing no even cycles.

1 Introduction

A *hypergraph* on a set X is a family $\mathcal{A}$ of labelled (but not necessarily distinct) subsets of X. According to Berge [1], a *cycle* in $\mathcal{A}$ is an alternating cyclic sequence $A_0, v_0, A_1, v_1, \ldots, A_{k-1}, v_{k-1}, A_0$ of distinct edges A_i of $\mathcal{A}$ and distinct vertices v_i of $\mathcal{A}$ such that $v_i \in A_i \cap A_{i+1}$ for all i modulo k. In this definition, we allow the members A_i of $\mathcal{A}$ to be equal as sets, but insist that they are distinct as members of $\mathcal{A}$. A cycle with k edges is referred to as a *k-cycle* or a *cycle of length k*.

Let us begin our discussion with the following well-known result: every maximal acyclic graph is a tree. We say that a hypergraph $\mathcal{A}$ is *acyclic* if it contains no cycle, and *connected* if for every nonempty subset e of X, $\mathcal{A} \cup \{e\}$ contains a cycle $\mathcal{C}$ with $e \in \mathcal{C}$. A *hypertree* is a connected acyclic hypergraph. The following holds (see Lovász [3]):

Theorem 1.1 *If $\mathcal{A}$ is an acyclic hypergraph on X, then $\sum_{A \in \mathcal{A}}(|A|-1) \leq |X|-1$, with equality if and only if $\mathcal{A}$ is a hypertree.*

In this paper, we are interested in the maximum size of a hypergraph containing no even cycle – in other words, no cycle of even size. Throughout the introduction, we assume $\mathcal{A}$ is a hypergraph on a set X. The *lower rank* of a hypergraph $\mathcal{A}$ is the size of a smallest element of A, namely $\min\{|A| : A \in \mathcal{A}\}$. Gyárfás, Jacobson, Kézdy and Lehel [5, 6] settled the extremal question for odd cycles by proving the following theorem:

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Theorem 1.2 *If $\mathcal{A}$ is a hypergraph of lower rank at least three, containing no odd cycle, then*

$$\sum_{A\in\mathcal{A}}(|A|-1)\leq 2|X|-2, \tag{1}$$

with equality if and only if $\mathcal{A}$ is the hypergraph on X consisting of two identical uniform hypertrees on X.

In other words, the extremal object $\mathcal{A}$ in Theorem 1.2 is a hypertree in which every edge is doubled – otherwise known as a *doubled hypertree.* In fact, the authors of [6] proved a stronger statement. They proved that if (1) is violated, then $\mathcal{A}$ contains a cycle $A_0, v_0, A_1, v_1, \ldots, A_{k-1}, v_{k-1}, A_0$ where some A_i contains at least three vertices in $\{v_0, , \ldots, v_{k-1}\}$. The main result of this paper is the solution of the extremal problem for even cycles:

Theorem 1.3 *Let $k \geq 2$, and let $\mathcal{A}$ be a hypergraph of lower rank at least k, containing no even cycle. Then $\sum_{A\in\mathcal{A}}(|A|-1) \leq \lfloor \frac{k}{k-1}(|X|-1)\rfloor - 1$.*

This bound is sharp. Note that the relations in Theorems 1.1 and 1.2 do not depend on the lower rank, while the relation in Theorem 1.3 does. The following bound is easily implied by Theorem 1.3:

Corollary 1.4 *Let $\mathcal{A}$ be a hypergraph on at least three vertices, containing no even cycle. Then $\sum_{A\in\mathcal{A}}(|A|-\frac{3}{2}) \leq \frac{3}{2}|X|-3$.*

The structure of the paper is as follows. The next section contains the notation required for the rest of the paper and the basic ideas of the proof of Theorem 1.3. As a forerunner for this proof, we give a new proof of Theorem 1.2 in the special case of triple systems. Our proof may be extended to all hypergraphs of lower rank at least three without too much difficulty. There we will also prove the theorem below:

Theorem 1.5 *Let $\mathcal{A}$ be a hypergraph containing no pair of cycles $\mathcal{C}$ and $\mathcal{C}'$ with $|\mathcal{C}| = |\mathcal{C}'|+1$. Then $\sum_{A\in\mathcal{A}}(|A|-2) \leq 2|X|-4$.*

In Section 3, we will prove Theorem 1.3, and give a construction to show that it cannot be improved.

2 Odd Cycles in Set Systems.

We write $G(A, B)$ to indicate that $G = G(A, B)$ is a bipartite graph with parts A and B.
In this section, we prove Theorems 1.2, in the case of triple systems, and Theorem 1.5. We will prove these theorems by appealing to the natural point-set incidence bipartite graph associated with a hypergraph $\mathcal{A}$ on X: this is the bipartite graph $G = G(\mathcal{A}, X)$ in which $x \in X$ is adjacent

to $A \in \mathcal{A}$ if $x \in A$. Conversely, we may associate to a bipartite graph $G(A, B)$ the hypergraph $\mathcal{A}$ on B consisting of the family of neighbourhoods of vertices in A. Therefore $G(\mathcal{A}, X)$ and $\mathcal{A}$ are equivalent representations of the same object.

It is clear that $\mathcal{A}$ contains a cycle of length k modulo two if and only if $G(\mathcal{A}, X)$ contains a cycle of length $2k$ modulo four. In this context, Theorem 1.2 on odd cycles may be stated in the following form:

Theorem 2.1 *Let $G = G(A, B)$ be a graph containing no cycle of length two modulo four, in which each vertex of A has degree at least three. Then G has size at most $|A| + 2|B| - 2$, with equality if and only if G is connected, no cut vertex is in A, and every block of G is a complete bipartite graph with two vertices in its smaller part.*

Note that the extremal configurations in Theorem 2.1 are precisely the point-set incidence bipartite graphs of doubled hypertrees. The proof we give below, in the case that all vertices of A have degree exactly three, can be extended a little to prove Theorem 2.1 in its full generality. Note that the restriction on the lower rank of the hypergraph (i.e the restriction on degrees in A in the incidence graph formulation) is required: a complete bipartite graph on X with $\lfloor |X|^2/4 \rfloor$ edges has no odd cycles, and Theorem 1.2 certainly does not apply. Before the proof, we require a few preliminaries:

Notation. We consider a graph G on X as a 2-uniform hypergraph, and $|G|$ denotes the number of edges of G. For convenience, an edge $\{x, y\}$ is written xy. We write $V(G)$ for the (non-empty) *vertex set* of G. If W is a set of vertices of G, we write $G - W$ for the graph on $V(G) - W$ consisting of all edges of G that are disjoint from W. We write $\Gamma_G(W)$ for the *neighbourhood* of W, i.e. $N(W) - W$. Let $d_G(W) = |\Gamma_G(W)|$ denote the *degree* of W. For a subgraph H of G, by the *neighbourhood* $\Gamma_G(H)$ of H we mean $\Gamma_G(V(H))$. The *distance* between two vertices u, v of a graph G, denoted $\text{dist}_G(u, v)$, is the length of a shortest path in G between u and v. A *component* of G is a maximal connected subgraph of G. A *cut vertex* of G is a vertex v such that $G - \{v\}$ has more components than G, and a block of G is a maximal subgraph H of G such that H has no cutvertices. A *cut set* of G is a set S of vertices of G such that $G - S$ has more components than G. A *pendant vertex* is an $x \in X$ such that $d_G(x) = 1$, and a *pendant block* in G is a block of G containing at most one cutvertex of G.

The first lemma we require is elementary, and therefore its proof is omitted:

Lemma 2.2 *Let $t \geq 3$, and let $P_1, P_2, \ldots, P_t$ be internally disjoint paths with the same pair of endpoints in a bipartite graph G. If G contains no cycle of length two modulo four, then the paths $P_1, P_2, \ldots, P_t$ have the same even length modulo four.*

The second lemma required was proved by Bondy and Vince [2]. For the sake of completeness, we include the proof.

Lemma 2.3 *Let G be any 2-connected graph with at least three vertices. Then G contains an induced cycle C with at least one of the following properties:*

(1) *$G - V(C)$ consists of precisely one component, or*
(2) *All components of $G - V(C)$ have the same pair of neighbours on C.*

Proof. Choose an induced cycle $C \subset G$ so that some component $H \subset G - V(C)$ has as many vertices as possible. If (1) does not hold, then let $H' \neq H$ be a component of $G - V(C)$. Let $u, v \in \Gamma(H)$ and $x, y \in \Gamma(H')$. If $\{u, v\} \neq \{x, y\}$, then there is an x-y path P, internally vertex-disjoint from C, and vertex-disjoint from H. Select an x-y subpath Q of C containing at most one of u or v. Then

$$V(H) \cup \{x\} \subset V(G) - V(P \cup Q) \qquad \text{or} \qquad V(H) \cup \{y\} \subset V(G) - V(P \cup Q).$$

This contradicts the maximality of H. So $\{u, v\} = \{x, y\}$ and (2) holds. ∎

Proof of Theorem 2.1 Suppose $G = G(A, B)$ has at least $|A| + 2|B| - 2$ edges and no cycle of length two modulo four. We aim to show that $|G| = |A| + 2|B| - 2$, and G is the incidence graph of a doubled hypertree. Proceed by induction on $|A| + |B|$. If G contains a cutvertex $b \in B$, then we may write $G = G_1 \cup G_2$ where $V(G_1) \cap V(G_2) = \{b\}$ and $|G_i| \geq |A \cap V(G_i)| + |B \cap V(G_i)| - 2$ for some $i \in \{1, 2\}$, for example, for $i = 1$. By induction, G_1 is the incidence graph of a doubled hypertree. It follows that $|G_1| = |A \cap V(G_1)| + |B \cap V(G_1)| - 2$ and therefore $|G_2| = |G| - |G_1| = |A \cap V(G_2)| + |B \cap V(G_2| - 2$. By induction, G_2 is also the incidence graph of a doubled hypertree, and therefore so is G.

Suppose, for a contradiction, that G contains no cutvertex in B. By induction, we may assume G is connected and $d_G(b) \geq 2$ for all $b \in B$. Then G has a pendant block F with at least three vertices. By Lemma 2.3, F contains a cycle C satisfying (1) or (2). For a subgraph H of G, let $A_H = A \cap V(H)$ and $B_H = B \cap V(H)$. Let H be a component of $F - V(C)$. By Lemma 2.2, $\Gamma_G(H) \subset A$ or $\Gamma_G(H) \subset B$, otherwise G contains a cycle of length two modulo four. Since $d_F(a) = 3$ for all but at most one $a \in A_F$, H satisfies (1) in Lemma 2.3. It follows that $d_F(b) = 2$ for every $b \in B_F$ and $\Gamma_G(H)$ contains all but at most one vertex of A_F. Note that if $\Gamma_G(H) = A_F - \{a\}$, then $a \in A$ is a cutvertex of G. Now H has size at least $|A_H| + 2|B_H| - 2$, and equality must hold, or G contains a cycle of length two modulo four. If $|A_H| = 1$, then $|H| = 0$, $|V(H)| = 1$, and F clearly contains a cycle of length six, a contradiction. If $|A_H| > 1$ then, by induction, H is the incidence graph of a doubled hypertree. Let $a, a' \in \Gamma_G(H)$. Then there are two paths of consecutive even lengths in H between a and a' and, together with a subpath of C, one of these paths gives a cycle of length two modulo four in F. This contradiction completes the proof of Theorem 2.1. ∎

Bondy and Vince used Lemma 2.3 to prove, by induction, that if G is a graph of minimum degree at least three, then G contains two cycles of consecutive lengths or of consecutive even lengths. A particular case of their result is as follows:

Proposition 2.4 *Let G be a bipartite graph of minimum degree at least three. Then G contains two cycles of consecutive even lengths.*

The hypergraph counterpart of Proposition 2.4 is:

Corollary 2.5 *Let $\mathcal{A}$ be a hypergraph of minimum degree and lower rank at least three. Then $\mathcal{A}$ contains a pair of cycles of consecutive lengths.*

To prove Theorem 1.5, the following incidence graph formulation must be proved: if $G = G(A, B)$ is a bipartite graph of size at least $2|A| + 2|B| - 3$, then G contains cycles of consecutive even lengths. In order to apply Proposition 2.4, we require one further lemma:

Lemma 2.6 *Let $G = G(A, B)$ be a bipartite graph of size at least $2|A|+2|B|-3$ with $|A|+|B| \geq 3$. Then G contains a subgraph of minimum degree at least three.*

Proof. One can check that each bipartite graph on n vertices with $3 \leq n \leq 5$ has fewer than $2n - 3$ edges. Let $|A| + |B| \geq 6$. If G itself has minimum degree at least three, the proof is complete. Otherwise let v_1 be a vertex of G of degree at most two, and let $G_1 = G - \{v_1\}$. In general, if G_i has minimum degree at least three, we are done, so we suppose there is a vertex $v_{i+1} \in G_i$ of degree at most two in G_i, and set $G_{i+1} = G_i - \{v_{i+1}\}$. Now if $t = |A| + |B| - 6$, then G_t has six vertices and size at least $2|A| + 2|B| - 3 - 2|A| + 2|B| + 12 = 9$. It follows that $G_t = K_{3,3}$, which has minimum degree at least three, as required. ∎

Proof of Theorem 1.5 Let $G = G(A, B)$ be a bipartite graph of size at least $2|A| + 2|B| - 3$. By Lemma 2.6, G contains a subgraph G' of minimum degree at least three. By Proposition 2.4, G' contains two cycles of consecutive even lengths, as required. ∎

3 Even Cycles in Set Systems.

In the last section, we described the correspondence between cycles in hypergraphs and in their natural point-line incidence graph representation. Using this description, we conclude that Theorem 1.3 is equivalent to the following statement:

Theorem 3.1 *Let $k \geq 2$, and let $G = G(A, B)$ be a bipartite graph containing no cycle of length zero modulo four, and in which every vertex of A has degree at least k. Then $|G| \leq |A| + \lfloor \frac{k}{k-1}(|B| - 1) \rfloor - 1$.*

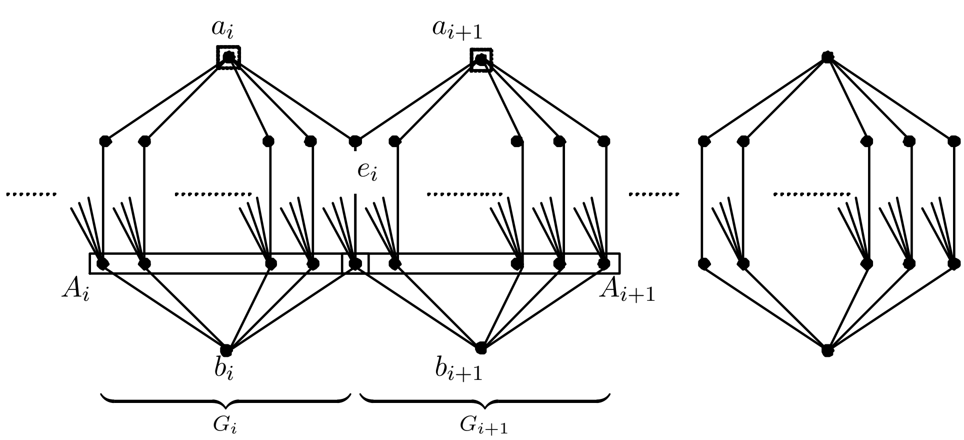
Construction. The following construction (see the illustration) shows that Theorems 3.1 and 1.3 cannot be improved: let m and $k \geq 2$ be positive integers, define a bipartite graph $H' = H'(m, k) = \bigcup_{i=1}^{m} G_i$, where $G_i = G_i(A_i, B_i)$ consists of k internally disjoint paths of length three between two vertices $a_i \in A_i$ and $b_i \in B_i$, where

- for $i = 1, \ldots, m-1$, there is an edge e_i, disjoint from $\{a_i, a_{i+1}, b_i, b_{i+1}\}$, with $G_i \cap G_{i+1} = \{e_i\}$.
- for $|i-j| \geq 2$, $V(G_i) \cap V(G_j) = \emptyset$.

We now add $k-3$ or $k-2$ pendant vertices in H', adjacent to each vertex of each A_i, in such a way that every vertex of $A = \bigcup A_i$ has degree exactly k, to obtain a bipartite graph $H_{m,k} = H_{m,k}(A, B)$. Then $H_{m,k}$ contains no cycle of length zero modulo four. If $m = 1$, then this obvious. Suppose that $H_{m-1,k}$ contains no cycle of length zero modulo four and C is such a cycle in $H_{m,k}$. Then C has some vertices outside of G_1 and some vertices in $G_1 - e_1$. Therefore, C contains both ends of e_1 and can be split into two paths connecting them. By the choice of m, each of these paths has length one modulo four. Hence C has length two modulo four, a contradiction to the choice of m. Moreover $|H_{m,k}| = mk^2 + k$, $|A| = mk + 1$, and $|B| = (k-1)^2 m + k$. It follows that

$$|A| + \lfloor \frac{k}{k-1}(|B|-1) \rfloor = mk + 1 + \frac{k}{k-1}[(k-1)^2 m + k - 1] = mk^2 + k + 1 = |H_{m,k}| + 1.$$

This completes the construction. An illustration is provided below.



To prove Theorem 3.1, the following analogue of Lemma 2.2 will be used.

Lemma 3.2 *Let $t \geq 2$, and let $P_1, P_2, \ldots, P_t$ be internally disjoint paths with the same pair of endpoints in a bipartite graph G. If G contains no cycle of length zero modulo four, then $|P_1| = |P_2| = \cdots = |P_t| = 1$ modulo four or $|P_1| = |P_2| = \cdots = |P_t| = 3$ modulo four, or $t = 2$ and P_1 and P_2 have different even lengths modulo four.*

An *ear* in a graph H is an inclusion maximal path whose all internal vertices have degree two. A ear is *non-trivial* if it has internal vertices (i.e. it has at least three vertices). In general, we represent paths by sequences of vertices, for example $(a_1, a_2, a_3, \ldots, a_k)$ is a path with endvertices a_1 and a_k.

Lemma 3.3 *Let H be a simple graph, not containing a subdivision of K_4, and with minimum degree at least two. If H is not a cycle, then it has at least two non-trivial ears.*

Proof. It is known that if H contains K_4 as a minor, then H contains a subdivision of K_4. The following fact is also known (see, for example, page 218 in West [4]): every simple graph on at least two vertices with at most one vertex of degree less than three contains a subdivision of K_4. Therefore H has at least one non-trivial ear, say, $Y = (v_0, v_1, \dots, v_m)$. When we contract all vertices $v_1, \dots, v_{m-1}$ into a new vertex v^*, the resulting graph H^* still satisfies the conditions of our claim: H^* is simple, contains no subdivision of K_4, and has no pendant vertices. Thus, using the fact above, H^* has a vertex of degree two distinct from v^*, and that vertex must belong to a ear in H distinct from Y. ∎

Proof of Theorem 3.1. Let $k \geq 2$, and let $G(A, B)$ be a minimal counterexample to Theorem 3.1. Then $G = G(A, B)$ has size at least $\phi_k(A, B) = |A| + \lfloor \frac{k}{k-1}(|B| - 1) \rfloor$ and no cycle of length zero modulo four. If $|A| = 1$, then $|G| = |B| < \phi_k(A, B)$, a contradiction. So $|A| > 1$. We proceed by a series of claims.

Claim 1 *The graph $G - \{b\}$ is connected for all $b \in B$. If $d_G(b) = 1$ for some $b \in B$, then the unique neighbour a of b has degree k in G.*

Proof. Assume that some $b \in B$ is a cut vertex. Let $G_1 = G_1(A_1, B_1)$ and $G_2 = G_2(A_2, B_2)$ be two connected subgraphs of G with at least two vertices, each having only the vertex b in common and whose union is G. As G_1 and G_2 are both subgraphs of G, neither G_1 nor G_2 has a cycle of length zero modulo four. By the minimality of G, $|G_i| < \phi_k(A_i, B_i)$, and

$$|G| = |G_1| + |G_2| < \phi_k(A_1, B_1) + \phi_k(A_2, B_2) = \phi_k(A, B).$$

This is a contradiction. Finally, if the last part of the claim were false for some $b \in B$, then $G - \{b\}$ would be a smaller counterexample than G, a contradiction. This proves Claim 1.

Edges e, f in a graph G are said to be *parallel* if they are equal as sets: in other words, they join the same pair of vertices. An edge of G is a *parallel edge* if there exists another edge of G to which it is parallel.

Claim 2 *If two vertices $b_1, b_2 \in B$ form a cut set in G, then*

(a) *$G - \{b_1, b_2\}$ has exactly two components*
(b) *the distance in G between b_1 and b_2 is two*
(c) *one of the components of $G - \{b_1, b_2\}$ is a tree consisting of a vertex a_0 in A joined to $k - 2$ pendant vertices of G.*

Proof. Assume that $\{b_1, b_2\} \subset B$ is a cut set in G. Let $H_1, \dots, H_m$ be the components of $G - \{b_1, b_2\}$. By Claim 1, each of H_i contains a b_1-b_2 path, P_i, for $i = 1, \dots, m$. Since G has no cycle of length zero modulo four, and $|P_i|$ is even for all i, the lengths of $P_1, \dots, P_m$ are distinct modulo four. Thus $m = 2$ and we may assume that $|P_1| = 0$ (mod 4) and $|P_2| = 2$ (mod

4). In particular, this proves (a). In the remainder of this claim, H_1 and H_2 denote the two components of $G - \{b_1, b_2\}$.

To prove (b) suppose, for a contradiction, that $\mathrm{dist}_G(b_1, b_2) \geq 4$. Let $H_2' = H_2'(C_2, D_2)$ be obtained from $G - H_1$ by identifying b_1 and b_2. Now H_2' contains no cycle of length zero modulo four (otherwise, G would also have such a cycle) and no parallel edges, since $\mathrm{dist}_G(b_1, b_2) \geq 4$. By the minimality of G, $|H_2'| < \phi_k(C_2, D_2)$. Consider now $H_1'(C_1, D_1) = G - H_2$. Again, $|H_1'| < \phi_k(C_1, D_1)$. Thus,

$$\begin{aligned} |G| = |H_1'| + |H_2'| &< \phi_k(C_1, D_1) + \phi_k(C_2, D_2) \\ &= |C_1| + |C_2| + \left\lfloor \frac{k(|D_1| - 1)}{k-1} \right\rfloor + \left\lfloor \frac{k(|D_2) - 1)}{k-1} \right\rfloor \\ &\leq |A| + \left\lfloor \frac{k}{k-1}(|B| - 1) \right\rfloor = \phi_k(A, B). \end{aligned}$$

This contradiction proves (b).

Finally, we prove (c). Let a_0 be a common neighbor of b_1 and b_2. Since G contains no cycle of length zero modulo four, there is only one such vertex a_0 and, by construction, it belongs to H_2. If (c) does not hold, then H_2 contains a vertex b_0 at distance three from a_0. In this case, we define $H_2'' = H_2''(C_2, D_2)$ as follows:

(i) in $G - H_1$, identify b_1 and b_2 into a new vertex b^*;
(ii) delete one of the parallel edges between a_0 and b^*;
(iii) add a path (a_0, b_0', a_0', b_0), where b_0' and a_0' are new vertices, and $k - 2$ new pendant neighbors $b_1', \ldots, b_{k-2}'$ of a_0'.

Let us check that H_2'' satisfies the requirements of Theorem 3.1. Clearly H_2'' contains no cycle of length zero modulo four (otherwise G would also have such a cycle), and H_2'' contains no parallel edges, by (ii). Finally, the degree of every vertex in C_2 is at least k, as we corrected the degree of a_0 in (iii). Now H_1 contains at least two vertices in A, since the shortest path from b_1 to b_2, with all internal vertices in H_1 has length at least four. Therefore $|V(H_2'')| < |V(G)|$. Let $H_1' = H_1'(C_1, D_1) = G - H_2$. Then we also have $|V(H_1')| < |V(G)|$. Hence, by the minimality of G, $|H_1'| < \phi_k(C_1, D_1)$ and $|H_2''| < \phi_k(C_2, D_2)$. Therefore, with $d_1 = |D_1|$ and $d_2 = |D_2|$,

$$\begin{aligned} |G| = |H_1'| + |H_2''| - k &< \phi_k(C_1, D_1) + \phi_k(C_2, D_2) - k \\ &= |A| + 1 + \left\lfloor \frac{k(d_1 - 1)}{k-1} \right\rfloor + \left\lfloor \frac{k(d_2 - 1)}{k-1} \right\rfloor - k \\ &\leq |A| + 1 + \left\lfloor \frac{k(d_1 + d_2 - 1)}{k-1} \right\rfloor - k - 1 \\ &= |A| - k + \left\lfloor \frac{k(|B| - 1 - (k-1))}{k-1} \right\rfloor = \phi_k(A, B). \end{aligned}$$

In the third line, we used the inequality $\lfloor a \rfloor + \lfloor b \rfloor \leq \lfloor a + b \rfloor$. This contradiction completes the proof of Claim 2.

Let G_1 be obtained from G by deleting from G all vertices of degree one in G. Suppose the parts of G_1 are A_1 and B_1.

Claim 3 *The minimum degree of G_1 is at least two.*

Proof. All remaining vertices in B did not change their degrees. Since $|A| > 1$, and G is connected, G_1 has no isolated vertices, by Claim 1. If a vertex $a \in A_1$ is a pendant vertex in G_1, then it has $k-1$ pendant neighbors, say, $b_1, \dots, b_{k-1}$, in G. Let $G'(A', B') = G - \{a, b_1, \dots, b_{k-1}\}$. By the minimality of G,

$$|G'| < \phi_k(A', B') = |A| - 1 + \left\lfloor \frac{k(|B| - 1 - (k-1))}{k-1} \right\rfloor < \phi_k(A, B)$$

and hence $|G| < \phi_k(A, B)$, a contradiction.

Claim 4 *No two vertices in A_1 of degree two in G_1 have a common neighbor of degree two.*

Proof. Assume that vertices a_1 and a_2 in A_1 of degree two have a common neighbor b_0 also of degree two. Let b_i, $i = 1, 2$ be the non-pendant neighbor of a_i distinct from b_0. Note that such vertices exist by Claim 1. Since G contains no cycle of length zero modulo four, $b_1 \neq b_2$. Then $\{b_1, b_2\}$ is a cut set in G. From Claim 2(b), we deduce that G_1 is a 6-cycle. It is easy to check that such a bipartite graph G satisfies the theorem. This completes the proof of Claim 4.

Let F be a pendant block in G_1. If F is a cycle, it has length at least six and therefore contains at least three vertices in A. This contradicts Claim 4. So there are at least two vertices of degree at least three in F. Let G_2 be the multigraph with parts A_2, B_2 obtained from F by replacing every nontrivial ear P of F with an edge connecting the end vertices of P. In other words, G_2 is the multigraph with no vertices of degree two such that F is a subdivision of G_2. Let G_3 be obtained from G_2 by replacing every set of parallel edges with a single edge. Since F had no cut vertices, neither do G_2 and G_3. As at least two vertices of F have degree at least three in F, $|V(G_3)| \geq 2$.

Claim 5 *Each parallel edge in G_2 corresponds to a path of length three in G.*

Proof. Suppose e_1 and e_2 are parallel edges in G_2 connecting vertices w and u. Since $|V(G_2)| \geq 2$ and G_2 is 2-connected, there is another path P connecting w and u in G_2. Then, by Lemma 3.2, e_1 and e_2 correspond to paths P_1 and P_2 in G having the same odd length modulo four. By Claim 4, neither of P_1 and P_2 has length five or more. As G is simple, P_1 and P_2 cannot both have length one. Therefore both paths have length three. This proves Claim 5.

An edge e of G_2 is called *i-complex* if e was obtained from a ear of length i in F. Then Claim 5 can be restated as follows: *every parallel edge in G_2 is 3-complex.* Let A^* be the (possibly empty) set of cutvertices of G_1 in F. If $A^* = \{a^*\}$, and a^* is an internal vertex of a nontrivial ear in F, then we denote by e^* the edge in G_2 corresponding to that ear, and let e^{**} be the edge in G_3 corresponding to the edge e^*. In this case, let $E^* = \{e^*\}$ and $E^{**} = \{e^{**}\}$, otherwise let $E^* = E^{**} = \emptyset$. A vertex $a \in A_2 - A^*$ is said to be *reducible* if all but one edge of $G_2 \backslash E^*$ incident with a is 3-complex.

Claim 6 *The multigraph G_2 contains no reducible vertices.*

Proof. Assume that $a_0 \in A_2$ is reducible. By definition, there must exist ears $P_1, P_2, \ldots, P_r \subset F$, each of length three, of the form

$$P_i = (a_0, b_i, a_i, b_i'),$$

where none of the a_i is in A^*. The neighbourhood of a_0 in G consists of the vertices $b_1, b_2, \ldots, b_r$, together with a set L_0 of pendant vertices on a_0 and, possibly, a vertex b_0 which may be pendant in G or in an ear of length three. This additional vertex is allowed by the definition of reducibility of a_0. Let us suppose $|L_0| = s$. Now denote by L_i the set of pendant neighbors of a_i, $i = 1, \ldots, r$. By Claim 2, $|L_i| = k - 2$ for $i = 1, \ldots, r$. Consider the graph

$$G' = G - \{a_i \mid i = 0, 1 \ldots, r\} - \{b_i \mid j = 0, 1 \ldots, r\} - \bigcup_{i=0}^{r} L_i.$$

with parts A' and B'. Since the neighborhood of each $a \in A'$ is the same in G' as it is in G, and G' contains no cycle of length zero modulo four, $|G'| < \phi_k(A', B')$. Note that $|A'| = |A| - r - 1$, $|B'| = |B| - r(k-1) - s$, and $|G'| = |G| - rk - r - s - 1$. Hence

$$\begin{aligned}
|G| &< \phi_k(A', B') + rk + r + s + 1 \\
&\leq \phi_k(A, B) - r - 1 - \left\lfloor \frac{k(r(k-1)+s)}{k-1} \right\rfloor + rk + r + s + 1 \\
&\leq \phi_k(A, B) - \left\lfloor \frac{s}{k-1} \right\rfloor \leq \phi_k(A, B).
\end{aligned}$$

This contradicts the definition of G. Thus Claim 6 is proved.

Claim 7 $|V(G_3)| > 2$.

Proof. If $|V(G_3)| = 2$, then G_2 forms a set of $l \geq 3$ paths $P_1, \ldots, P_l$ between two vertices, say v_1 and v_2. By Claim 5, the length of every P_i is three. Thus one of v_1 and v_2, say, v_1, is in A_2, and, moreover, v_1 is reducible. This contradicts Claim 6.

As $|A \cup B| > 2$, and G_3 has no cut-vertices, G_3 is 2-connected. By Claim 5, G_3 has vertices of degree two.

Claim 8 *No $a_0 \in A_3$ has degree two in G_3.*

Proof. Suppose that $a_0 \in A_3$ has degree two in G_3 and its neighbors are b_1 and b_1'. Then, unless a_0 is incident with an edge of E^* and an edge of G_2 which is not a parallel edge in G_2, a_0 is reducible, contradicting Claim 7. So a_0 is incident in G_2 to a nonparallel edge, say $\{a_0, b_1'\}$, and to e^*. Since $d_{G_2}(a_0) \geq 3$, the edge $e^* = \{a_0, b_1\}$ is parallel in G_2. By Claim 5, $b_1 \in B$. Let b_2 be the first vertex on the path in G corresponding to the edge $\{a_0, b_1'\}$ in G_2. Note that $b_1' = b_2$ is possible if the path consists only of $\{a_0, b_1'\}$. In any case, $\{b_1, b_2\}$ is a cut set in G. Then Claim 3 implies that G_3 has only two vertices, namely a_0 and b_1. This contradicts Claim 7, and proves Claim 8.

Now we are ready to prove Theorem 3.1. We consider two cases: (1) G_3 is a cycle and (2) G_3 is not a cycle. In case (1), the degree of each vertex in G_2 is at least three, by definition of G_2, and each edge of G_3 is a parallel edge in G_2. By Claim 5, the ends of a parallel edge in G_2 belong to distinct parts of G. This violates Claim 8, and completes the proof in case (1).

Suppose case (2) arises. By Claim 8, no vertex of A_2 has degree two in G_3. If G_3 has no nontrivial ears, then G_3 contains a subdivision H of K_4 and so G contains H. Take two branching vertices v_1, v_2 of H (in other words, vertices of degree three in H) from the same part of G. By Lemma 3.2, G contains a cycle of length zero modulo four. Therefore G_3 has at least one nontrivial ear, containing a vertex $b_1 \in B_2$ of degree two, such that if E^* consists of the parallel edge e^* in G_2, then b_1 is not incident in G_3 with $e^{**} \in E^{**}$.

Let v_1 and v_2 be the neighbors of b_1 in G_2. Then one of the edges $\{b_1, v_1\}$ and $\{b_1, v_2\}$, for instance $\{b_1, v_1\}$, is a parallel edge in G_2 and hence the edges parallel to $\{b_1, v_1\}$ are 3-complex. In particular, $v_1 \in A_2$. Let a_0 be the first vertex on the ear corresponding to $\{b_1, v_2\}$ in G_1. Now there are no two internally disjoint paths connecting v_1 and a_0 in $G_2 - b_1$. Let b_2 be the closest vertex (in G) to v_1, such that a_0 and v_1 are in different components of $(G_1 - b_1) - b_2$. Since b_2 is the closest such vertex, either it is adjacent to v_1, or there are two internally disjoint paths from v_1 to b_2 in $G - \{b_1\}$. In both cases, $b_2 \in B$. Then $\{b_1, b_2\}$ forms a cut set in G so, by Claim 5, a_0 is a common neighbor of b_1 and b_2 and has degree two in G_1. Observe that $d_G(v_1) = k$, otherwise we could delete an ear from v_1 to b_1 reducing both $|G|$ and $\phi_k(A, B)$ by exactly $k + 1$. Let the ears in G_1 from v_1 to b_1 be denoted

$$P_i = (v_1, b_i', a_i', b_1), i = 1, \ldots, r.$$

Since $d_G(v_1) = k$, we have $r \leq k - 1$. Let $G' = G'(A', B')$ be obtained from G by deleting the vertices $a_1', \ldots, a_r', a_0$, all their pendant neighbors, and b_1. Then $|G| - |G'| = (r + 1)k$ and

$$\begin{aligned}
\phi_k(A,B) - \phi_k(A',B') &= r + 1 + \Big\lfloor \frac{k(|B|-1)}{k-1} \Big\rfloor - \Big\lfloor \frac{k(|B| - (k-2)(r+1) - 2)}{k-1} \Big\rfloor \\
&\geq r + 1 + \Big\lfloor \frac{k(1 + (k-2)(r+1))}{k-1} \Big\rfloor \\
&= r + 1 + (1 + (k-2)(r+1)) + (r+1) - \Big\lceil \frac{r}{k-1} \Big\rceil \geq (r+1)k.
\end{aligned}$$

In the last line, we used $r \leq k - 1$ and the inequality $\lfloor a + b \rfloor \geq \lfloor a \rfloor + \lfloor b \rfloor$. This contradicts the minimality of G. The proof of Theorem 3.1 is now complete.

We now turn to Corollary 1.4. The incidence graph version of it is:

Corollary 3.1 *Let G be a bipartite graph on at least 4 vertices, containing no cycle of length zero modulo four. Then $|G| \leq \lfloor 3n/2 \rfloor - 3$.*

Proof. If $n \geq 4$, then $\lfloor 3n/2 \rfloor - 2 \geq n$. This proves the corollary for $n \leq 5$. Let G be a counterexample to the corollary with fewest vertices, and parts A and B such that $|A \cup B| =$

$n > 5$. Clearly, we may assume that the minimum degree of G is at least two, and $|A| \geq |B|$. Then by Theorem 3.1 with $k = 2$,

$$|G| \leq \phi_2(A, B) - 1 = |A| + 2|B| - 3 \leq \lfloor 3n/2 \rfloor - 3,$$

a contradiction. ∎

This corollary is best possible for all $n \geq 4$. Indeed, for even n, let $G_n = G_n(A, B)$ consist of $n/2 - 1$ internally disjoint paths of length 3 between two fixed vertices. Then $|A| = |B|$, $|G_n| = 3(n/2 - 1) = \lfloor 3n/2 \rfloor - 3$, and G_n only contains cycles of length six. For an odd $n \geq 5$, G_n is obtained from G_{n+1} by deleting a vertex of degree two.

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