Recurrence Equations

1 Fibonacci Numbers

A population of creatures starts off with one creature. The rule of growth of the population is this: immediately after two time steps, a creature gives birth to a new creature, and then gives birth to one creature immediately after every time step thereafter. The aim is to determine at \( n \) steps in time steps what the population is. If \( F_n \) is the population at time \( n \), then \( F_1 = 1 \), \( F_2 = 1 \), \( F_3 = 2 \), \( F_4 = 3 \) and we could in theory work out \( F_n \) for any value of \( n \). The numbers \( F_n \) are called Fibonacci Numbers. The table below shows the population at each time step of each generation up to time ten:

<table>
<thead>
<tr>
<th>Time</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Original</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1st Generation</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2nd Generation</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

It is much more useful, however, to have a formula for \( F_n \). The first step in this direction is to note that

\[
F_n = F_{n-1} + F_{n-2}
\]

for all \( n > 2 \). This is true since every creature at time \( n = 2 \) gives birth to a new creature, whereas every creature at time \( n = 1 \) remains and does not give birth. So there are \( F_{n-2} \) new creatures and \( F_{n-1} \) creatures which do not give birth. If we repeat the formula, we get the nice formula

\[
F_n = F_{n-2} + F_{n-3} + \cdots + F_1
\]

for the population at time \( n \). Still this requires knowledge of \( F_{n-2}, F_{n-3}, \ldots, F_1 \). The main result on Fibonacci numbers is the following:

Theorem. Let \( \varphi = \frac{1}{2}(1 + \sqrt{5}) \) and \( \varphi = \frac{1}{2}(1 - \sqrt{5}) \). Then

\[
F_n = \left(\frac{1}{\sqrt{5}}\varphi^n - \frac{1}{\sqrt{5}}\overline{\varphi}^n\right).
\]

A function \( f(n) \) grows exponentially fast if there is a constant \( c > 1 \) such that \( f(n) > c^n \) for all \( n \). The Fibonacci Numbers, therefore, grow exponentially fast. In fact, \( F_n \) is the largest integer less than \( \frac{1}{\sqrt{5}}\varphi^n \), since \( \overline{\varphi}^n \) is extremely small if \( n \) is large.

2 Recurrence Equations

A recurrence equation for a sequence \((a_n)_{n \geq 1}\) is an equation in terms of \( a_n, a_{n-1}, \ldots, a_1 \). For example, \( a_n = a_{n-1} + a_{n-2} \) is a recurrence equation, and it defines the Fibonacci numbers. Other examples of recurrence equations are \( a_n = 2a_{n-1}, a_n = a_{n-1} + 1, a_n = a_{n-1} + 2a_{n-2} \) and \( a_n = n \sin(a_{n-1}) \). The general question here is how we solve such equations. First we need some initial conditions – these are prescriptions of the value of \( a_n \) for the first few
values of \( n \). For example, the equation \( a_n = 2a_{n-1} \) can’t be solved explicitly for \( n \geq 1 \) if we don’t know \( a_1 \). Let’s suppose \( a_1 = 2 \). Then \( a_2 = 4, a_3 = 8, a_4 = 16, \) and we can see the pattern giving \( a_n = 2^n \). However, in general it is not easy to see a pattern – for example \( a_n = n \sin(a_{n-1}) \) with \( a_1 = 1 \) does not have a nice pattern which allows us to guess the answer. So we need a general way to handle equations. We consider equations of the form 

\[
a_n + \alpha a_{n-1} + \beta a_{n-2} = 0
\]

where \( \alpha, \beta \) are numbers, and we are given the values of \( a_1 \) and \( a_2 \) (the initial conditions). So the Fibonacci equation fits into this framework, with \( \alpha = \beta = -1 \) and \( a_1 = a_2 = 1 \). The main theorem for solving these equations is as follows. To state the theorem, we need the notion of the characteristic equation: the characteristic equation of the recurrence \( a_n + \alpha a_{n-1} + \beta a_{n-2} = 0 \) is the quadratic equation \( x^2 + \alpha x + \beta = 0 \).

**Theorem.** Let \( A \) and \( B \) be distinct roots of the equation \( x^2 + \alpha x + \beta = 0 \). Then the solution to the recurrence equation \( a_n + \alpha a_{n-1} + \beta a_{n-2} = 0 \) with initial conditions \( a_1 = a \) and \( a_2 = b \) is

\[
a_n = \left( \frac{b - aA}{B(A - B)} \right) A^n + \left( \frac{b - Ba}{A(A - B)} \right) B^n.
\]

It is not important to remember the numbers \( \frac{b - aA}{B(A - B)} \) and \( \frac{b - Ba}{A(A - B)} \), since these are found by knowing that the solution is \( a_n = cA^n + dB^n \) for some constants \( c \) and \( d \), and then solving for \( c \) and \( d \) using \( a_1 = a \) and \( a_2 = b \). For example, the Fibonacci equation \( a_n = a_{n-1} + a - n - 2 \) has characteristic equation \( x^2 - x - 1 = 0 \). Using the quadratic formula, we get

\[
A = \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad B = \frac{1 - \sqrt{5}}{2}.
\]

Therefore we know \( a_n = cA^n + dB^n \) for some constants \( c \) and \( d \). Now since \( a_1 = 1 = a_2 \), we know

\[
1 = cA + dB \quad \text{and} \quad 1 = cA^2 + dB^2
\]

which gives us \( c = 1/\sqrt{5} \) and \( d = -1/\sqrt{5} \), agreeing with the theorem on Fibonacci numbers.

Here is another example. Consider the equation \( a_n = 3a_{n-1} - 2a_{n-2} \) with initial conditions \( a_1 = 3 \) and \( a_2 = 5 \). The characteristic equation is \( x^2 - 3x + 2 = 0 \), so the roots are \( A = 1 \) and \( B = 2 \). By the theorem

\[
a_n = c + d2^n.
\]

Since \( a_1 = 3 \) and \( a_2 = 5 \) we get \( c + 2d = 3 \) and \( c + 4d = 5 \), giving \( c = d = 1 \). So \( a_n = 2^n + 1 \).

Recurrence equations are one of the fundamental tools of mathematicians, and they appear everywhere in combinatorics. In some sense they are discrete analogs of differential equations. An example of a famous recurrence equation is the equation for Catalan numbers, which amongst other things count the number of binary trees with \( n \) vertices, the number of bracketings of an \( n \)-variable formula, the number of triangulations of a polygon with \( n \) sides, and the ballot problem for \( n \) voters. The Catalan numbers are given by the recurrence

\[
(n + 2)C_{n+1} = 2(2n + 1)C_n
\]

where \( C_2 = 2 \). You can check by direct substitution that \( C_n = \frac{1}{n+1} \binom{2n}{n} \) solves this equation, although we omit the method that gives this answer.