Recurrence Equations

1 Fibonacci Numbers

A population of creatures starts off with one creature. The rule of growth of the population is this: immediately after two time steps, a creature gives birth to a new creature, and then gives birth to one creature immediately after every time step thereafter. The aim is to determine at n steps in time steps what the population is. If F_n is the population at time n, then $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$ and we could in theory work out F_n for any value of n. The numbers F_n are called *Fibonacci Numbers*. The table below shows the population at each time step of each generation up to time ten:

<table>
<tr><td>Time</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td><td>7</td><td>8</td><td>9</td><td>10</td></tr>
<tr><td>Original</td><td>|</td><td>|</td><td>|</td><td>|</td><td>|</td><td>|</td><td>|</td><td>|</td><td>|</td><td>|</td></tr>
<tr><td>1st Generation</td><td></td><td></td><td>|</td><td>||</td><td>|||</td><td>||||</td><td>|||||</td><td>||||||</td><td>|||||||</td><td>||||||||</td></tr>
<tr><td>2nd Generation</td><td></td><td></td><td></td><td></td><td>|</td><td>|||</td><td>||||</td><td>||||||</td><td>|||||||||||||</td><td>||||||||||||||||||||||</td></tr>
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It is much more useful, however, to have a formula for F_n. The first step in this direction is to note that

$$F_n = F_{n-1} + F_{n-2}$$

for all $n > 2$. This is true since every creature at time $n = 2$ gives birth to a new creature, whereas every creature at time $n = 1$ remains and does not give birth. So there are F_{n-2} new creatures and F_{n-1} creatures which do not give birth. If we repeat the formula, we get the nice formula

$$F_n = F_{n-2} + F_{n-3} + \cdots + F_1$$

for the population at time n. Still this requires knowledge of $F_{n-2}, F_{n-3}, \dots, F_1$. The main result on Fibonacci numbers is the following:

Theorem. *Let* $\varphi = \frac{1}{2}(1+\sqrt{5})$ *and* $\overline{\varphi} = \frac{1}{2}(1-\sqrt{5})$. *Then*

$$F_n = \frac{1}{\sqrt{5}}\varphi^n - \frac{1}{\sqrt{5}}\overline{\varphi}^n.$$

A function $f(n)$ grows *exponentially fast* if there is a constant $c > 1$ such that $f(n) > c^n$ for all n. The Fibonacci Numbers, therefore, grow exponentially fast. In fact, F_n is the largest integer less than $\frac{1}{\sqrt{5}}\varphi^n$, since $\overline{\varphi}^n$ is extremely small if n is large.

2 Recurrence Equations

A recurrence equation for a sequence $(a_n)_{n\geq 1}$ is an equation in terms of $a_n, a_{n-1}, \dots, a_1$. For example, $a_n = a_{n-1} + a_{n-2}$ is a recurrence equation, and it defines the Fibonacci numbers. Other examples of recurrence equations are $a_n = 2a_{n-1}$, $a_n = a_{n-1} + 1$, $a_n = a_{n-1} + 2a_{n-2}$ and $a_n = n\sin(a_{n-1})$. The general question here is how we solve such equations. First we need some *initial conditions* – these are prescriptions of the value of a_n for the first few values of n. For example, the equation $a_n = 2a_{n-1}$ can't be solved explicitly for $n \geq 1$ if we don't know a_1. Let's suppose $a_1 = 2$. Then $a_2 = 4$, $a_3 = 8$, $a_4 = 16$, and we can see the pattern giving $a_n = 2^n$. However, in general it is not easy to see a pattern – for example $a_n = n \sin(a_{n-1})$ with $a_1 = 1$ does not have a nice pattern which allows us to guess the answer. So we need a general way to handle equations. We consider equations of the form $a_n + \alpha a_{n-1} + \beta a_{n-2} = 0$ where α, β are numbers, and we are given the values of a_1 and a_2 (the initial conditions). So the Fibonacci equation fits into this framework, with $\alpha = \beta = -1$ and $a_1 = a_2 = 1$. The main theorem for solving these equations is as follows. To state the theorem, we need the notion of the *characteristic equation*: the characteristic equation of the recurrence $a_n + \alpha a_{n-1} + \beta a_{n-2} = 0$ is the quadratic equation $x^2 + \alpha x + \beta = 0$.

Theorem. *Let A and B be distinct roots of the equation $x^2 + \alpha x + \beta = 0$. Then the solution to the recurrence equation $a_n + \alpha a_{n-1} + \beta a_{n-2} = 0$ with initial conditions $a_1 = a$ and $a_2 = b$ is*

$$a_n = \Big(\frac{b - aA}{B(A-B)}\Big)\, A^n \quad + \quad \Big(\frac{b - Ba}{A(A-B)}\Big)\, B^n.$$

It is not important to remember the numbers $\frac{b-aA}{B(A-B)}$ and $\frac{b-Ba}{A(A-B)}$, since these are found by knowing that the solution is $a_n = cA^n + dB^n$ for some constants c and d, and then solving for c and d using $a_1 = a$ and $a_2 = b$. For example, the Fibonacci equation $a_n = a_{n-1} + a - n - 2$ has characteristic equation $x^2 - x - 1 = 0$. Using the quadratic formula, we get

$$A = \frac{1+\sqrt{5}}{2} \qquad \text{and} \qquad B = \frac{1-\sqrt{5}}{2}.$$

Therefore we know $a_n = cA^n + dB^n$ for some constants c and d. Now since $a_1 = 1 = a_2$, we know

$$1 = cA + dB \qquad \text{and } 1 = cA^2 + dB^2$$

which gives us $c = 1/\sqrt{5}$ and $d = -1/\sqrt{5}$, agreeing with the theorem on Fibonacci numbers.

Here is another example. Consider the equation $a_n = 3a_{n-1} - 2a_{n-2}$ with initial conditions $a_1 = 3$ and $a_2 = 5$. The characteristic equation is $x^2 - 3x + 2 = 0$, so the roots are $A = 1$ and $B = 2$. By the theorem

$$a_n = c + d2^n.$$

Since $a_1 = 3$ and $a_2 = 5$ we get $c + 2d = 3$ and $c + 4d = 5$,] giving $c = d = 1$. So $a_n = 2^n + 1$.

Recurrence equations are one of the fundamental tools of mathematicians, and they appear everywhere in combinatorics. In some sense they are discrete analogs of differential equations. An example of a famous recurrence equation is the equation for Catalan numbers, which amongst other things count the number of binary trees with n vertices, the number of bracketings of an n-variable formula, the number of triangulations of a polygon with n sides, and the ballot problem for n voters. The Catalan numbers are given by the recurrence

$$(n+2)C_{n+1} = 2(2n+1)C_n$$

where $C_2 = 2$. You can check by direct substitution that $C_n = \frac{1}{n+1}\binom{2n}{n}$ solves this equation, although we omit the method that gives this answer.